Chapter 6.3-6.4: Combinations and Permutations

Wednesday, October 14

Summary

- Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- Binomial Theorem: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$
- Multinomial Theorem: $(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots n_m = n} \frac{n!}{n_1! n_2! \cdots n_m!} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$

Warmup

1. Six people are in a club. How many ways to choose three executive members? How many ways are there to choose a president and two vice presidents? Why is the second number divisible by the first?

   There are $\binom{6}{3} = 20$ ways to pick out three members, and $\binom{6}{2}\binom{4}{1} = 60$ ways to pick out a prez and two VPs. The first number divides the second because we can make the first process a subprocess of the second— pick three people, then pick 1 of those three to be the president and make the other two VPs.

2. Ten people are in a room, and everyone shakes everyone else's hand exactly once. How many handshakes were there?

   There were $\binom{10}{2} = 45$ handshakes.

3. Seven horses run a race. Assuming there are no ties, how many ways are there to hand out gold, silver, and bronze medals?

   There are $7 \cdot 6 \cdot 5 = 210$ ways to hand out the medals.

4. How many ways to hand out three medals if all of the medals are identical?

   There would be $\binom{7}{3} = 35$ ways to hand out three medals.

5. How many ways to hand out 1 gold, 1 silver, 1 bronze, and 4 participation medals?

   Still 210. . . once the gold, silver, and bronze have been handed out there is only 1 way to hand out the participation medals.

Binomial and Multinomial Coefficients

1. How many distinct ways are there to arrange the letters in COUSCOUS?

   There are 8! ways to rearrange the letters, then divide by 2^4 for the 4 pairs of letters (each letter can be swapped with its partner to get an identical arrangement), so $\frac{8!}{2^4} = 2520$ ways in all.

2. How many distinct ways are there to arrange the letters in MISSISSIPPI?

   The number of ways is $\binom{11}{1,2,4,4} = 34650$.

3. If I flip a coin 7 times, how many ways are there to get an even number of heads?

   The number of ways to get an even number of heads is $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 1 + 21 + 35 + 7 = 64$. Note that this is exactly half of the total number of outcomes.

4. If I roll five dice, how many ways are there to get 3 of one number and 2 of another?

   There are 6 ways to pick the number that gets rolled 3 times and 5 ways to pick the number that gets rolled 2 times, so 30 in all.

5. If I draw 5 cards from a deck, how many ways are there (ignoring the order in which the cards are drawn) to draw 3 cards of one value and 2 of another?

   There are 13 ways to pick the value that gets drawn 3 times and 12 ways to pick the value that gets drawn twice, so 156 in all.

Combinatorial Proofs

1. Show that if n is a positive integer then $\binom{2n}{2} = 2\binom{n}{2} + n^2$, by combinatorial proof and by algebraic manipulation. (Hint: there are n boys and n girls. If you want to pick 2 people for a team, break down by the number of girls you pick.)

   Combinatorial: If there are n boys and n girls and you want to pick 2 of them (so, $\binom{2n}{2}$ options), there are $\binom{n}{2}$ ways to pick 2 boys and $\binom{n}{2}$ ways to pick 2 girls and n^2 ways to pick 1 boy and 1 girl.

   Algebraic: $\binom{2n}{2} = \frac{2n(2n-1)}{2} = 2n^2 - n = n^2 + (n^2 - n) = n^2 + 2\binom{n}{2}$.

Pascal's Triangle

1. Prove the *hockeystick identity*

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

   using Pascal's identity (plus induction! For the induction, fix n arbitrarily and then use induction on r).

   Fix n arbitrarily. For $r = 0$ this is just $\binom{n}{0} = \binom{n+1}{0}$, which is true since $\binom{m}{0} = 1$ for any m.

   For the inductive step: suppose that the formula holds for r, then

$$\begin{aligned}\sum_{k=0}^{r+1} \binom{n+k}{k} &= \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+2}{r+1}\end{aligned}$$

   So if the formula holds for r then it also holds for $r+1$. Since n was picked arbitrarily, the formula works for any n and $r \geq 0$.

2. Illustrate the above identity on Pascal's Triangle, and use it to prove that $1+2+\cdots+n=\frac{n(n+1)}{2}$.

   On Pascal's Triangle it looks like a hockey stick— a line going down on a diagonal in one direction ending with a small hook in the other direction. Applied to $n=1$, $r=(m-1)$, this gives $\sum_{k=0}^{m}\binom{k+1}{k}=\binom{m+1}{m-1}$. Using the fact that $\binom{a}{b}=\binom{a}{a-b}$, this means that $\sum_{k=0}^{m}k=\sum_{k=0}^{m}\binom{k+1}{1}=\binom{m+1}{2}=\frac{m(m+1)}{2}$.