Chapter 6.1-6.2: Counting

Monday, October 12

Warmup

1. Define a sequence a_n by $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, $a_n = 2 \cdot a_{n-1} - a_{n-2}$. Find a non-recursive formula for a_n and prove that it is correct.

   $a_n = n$. Proof by induction: $n_0 = 0$ and $n_1 = 1$ for the base case.

   Then if the formula works for $n-1$ and n, $a_{n+1} = 2 \cdot a_n - a_{n-1} = 2n - (n-1) = n+1$. This completes the proof by induction.

2. How many numbers between 1 and 60 are divisible by 2? 30

3. How many are divisible by 3? 20

4. How many are divisible by 5? 12

5. How many are divisible by 2 or 3 or 5? By inclusion-exclusion, the number is $30+20+12-10-6-4+2 = 44$.

Inclusion-Exclusion

1. How many numbers between 1 and 1000

   (a) Are divisible by both 7 and 11?
   $\lfloor 1000/7 \rfloor = 142$ numbers are divisible by 7, $\lfloor 1000/11 \rfloor = 90$ are divisible by 11, and $\lfloor 1000/77 \rfloor = 12$ are divisible by 77 (these are the only ones divisible by 7 and 11).

   (b) Are divisible by either 7 or 11?
   Using inclusion-exclusion, the number divisible by 7 or 11 is 142 + 90 - 12 = 220.

   (c) Are divisible by 7 but not by 11?
   The number divisible by 7 but not 11 is 142 - 12 = 130 (draw a Venn diagram to justify this).

   (d) Are divisible by neither 7 nor 11?
   This is the same as NOT (divisible by 7 or 11), so the number of such elements is 1000 - 220 = 780.

2. 36 students go to a hot dog stand and order hot dogs. Every student orders at least one topping. You have the following information about their topping choices:

   (a) 18 ask for mustard.
   (b) 21 ask for onions.
   (c) 18 ask for relish.
   (d) 8 ask for mustard but not onions.
   (e) 31 ask for onions or relish (or both).
   (f) 17 ask for exactly two toppins.
   (g) 2 ask for all three toppings.

   How many students order exactly one topping? (Try making a Venn diagram.)

   Denote the numbers of students who get mustard ONLY, onion ONLY, and relish ONLY by M, O, and R, respectively. Those who get two ingredients are denoted by MO, MR, and OR, and those who get three by MOR. Then we are given the equations

   (a) M + MO + MR + MOR = 18
   (b) O + MO + OR + MOR = 21

(c) R + MR + OR + MOR = 18

(d) M + MR = 8

(e) O + R + MO + MR + OR + MOR = 31

(f) MO + MR + OR = 17

(g) MOR = 2

(h) M + O + R + MO + MR + OR + MOR = 36

Combining (5) and (8) gives M = 5, and combining (5), (6), and (7) gives O + R = 12. Therefore M + O + R = 17 students get exactly one ingredient.

Pigeonhole Principle

1. It is said of the town of Lake Wobegon that "all the women are strong, all the men are good looking, and all the children are above average." Discuss.

   Sorry, not everyone can be above average.

2. Come up with a related formal statement and prove it. How does this relate to the Pigeonhole Principle?

   How about this: *For any set of n numbers, at least one of the numbers must be less than or equal to the average.*

   Proof by contradiction: Suppose $a_j > \frac{1}{n}\sum_{i=1}^{n} a_i$ for all $1 \leq j \leq n$. Then $\sum_{j=1}^{n} a_j > \sum_{j=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) = n \cdot \frac{1}{n}\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_i$, a contradiction.

   To relate this to the Pigeonhole Principle, think of the a_i as the number of pigeons that are in box i. At least one must be less than or equal to the average (and similarly, at least one must be greater than or equal to the average).

Tree Diagrams

1. Use a tree diagram to find the number of subsets of $\{3, 7, 9, 11, 24\}$ such that the sum of the elements in the subset is less than 28.

   There are at least two ways to make the decision tree: first, one can split by which number is the smallest element of the set (leading to 6 branches: none/3/6/9/11/24). Second, one could pick an element and ask whether it is in the set (2 branches: yes/no).

   For example: if 24 is in the set, then the sum of the other elements must be less than four. This leads to only two options, $\{3, 24\}$ and $\{24\}$.

   Otherwise, 24 is not in the set and we have the new problem of finding the number of subsets of $\{3, 7, 9, 11\}$ whose sum is less than 28. There is a shortcut here: the sum of all four elements is 30 (too large), but the sum of any three elements is at most 27. Therefore ANY subset except for the whole one will work, and there are $2^4 - 1 = 15$ such options.

   The total number of options is therefore 17.

Division Rule and Symmetries

1. How many ways are there to seat 5 people around a circular table?

   24: There are $5! = 120$ ways normally, divided by 5 ways to rotate the table once people have been seated.

2. If we make a 4-sided die out of a tetrahetron (4 faces, all equilateral triangles), then how many possible arrangements of the numbers are there?

   There are $4! = 24$ ways to place the numbers on the sides but 12 ways to rotate the tetrahedron, so there are only 2 disticnt arrangements in all. (Imagine the "1" on the bottom face. Then going around the top clockwise, the numbers must either read 2-3-4 or 2-4-3. These are the only options.)

3. How many possible arrangements are there on a 6-sided die?

   On the 6-sided die there are 24 symmetries (pick a face to be the top face, then spin it around 4 ways). The total number of distinct dice is therefore $6!/24 = 30$.