Chapter 5.1-5.3: More Induction and Recursion

Wednesday, October 7

Warmup

1. Prove: If $A_{n+1} \subseteq A_n$ for all $n \geq 1$ then $\bigcap_{i=1}^{n} A_i = A_n$.

2. What, if anything, is wrong with the following proof?

   Theorem 0.1 (Questionable Theorem) *Define f_n by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Then for all $n \geq 1$, $3f_n + f_{n+1} = 2 \cdot f_{n+2}$.*

   Proof: For the base case: $f_1 = 1, f_2 = 1$, and $f_3 = 2$, so when $n = 1$ we have $3f_n + f_{n+1} = 3f_1 + f_2 = 4 = 2 \cdot f_3 = 2 \cdot f_{n+2}$.

   Inductive step: Suppose that $3f_n + f_{n+1} = 2f_{n+2}$. Then

   $$\begin{aligned} 3f_{n+1} + f_{n+2} &= 3(f_n + f_{n-1}) + f_{n+1} + f_n \\ &= (3f_n + f_{n+1}) + (3f_{n-1} + f_n) \\ &= 2f_{n+2} + 2f_{n+1} \\ &= 2f_{n+3}. \end{aligned}$$

   By induction, $3f_n + f_{n+1} = 2f_{n+2}$ for all $n \geq 1$.

3. Which Fibonacci numbers are even? Come up with a conjecture and prove it.

Recursion

1. Define a sequence a_n by $a_0 = 1$, $a_1 = 3$ and $a_n = a_{n-1} + 2 \cdot a_{n-2}$ for $n \geq 2$. Find a_6. Prove that $a_n = \dfrac{2^{n+2} - (-1)^n}{3}$.
2. Define a sequence a_n by $a_0 = 1$, $a_1 = 3$, and $a_n = \sum_{i=1}^{n-1} a_i$ for $n \geq 2$. Find a formula for a_n and prove that it is correct.
3. Prove: $\gcd(f_{n+1}, f_n) = 1$ for all $n \geq 0$.
4. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for $n \geq 1$.
5. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ for $n \geq 1$.
6. (★) Prove that $\gcd(f_m, f_n) = f_{\gcd(m,n)}$ for any $m, n \in \mathbb{N}$.

Recursive Games

You and a friend are playing a game: there is a pile of stones. You take turns removing stones from the pile— during your turn, you may remove 1, 2, or 3 stones. Whoever removes the last stone wins.

1. Prove that the second player has a winning strategy if the pile begins with 8 stones.
2. Who has a winning strategy if the pile begins with 15 stones? 16 stones?
3. Suppose that the pile has n stones, and you may choose whether to go first or second. How should you decide?