Chapter 5.1-5.3: More Induction and Recursion

Wednesday, October 7

Warmup

1. Prove: If $A_{n+1} \subseteq A_n$ for all $n \geq 1$ then $\bigcap_{i=1}^{n} A_i = A_n$.

   Base case: when $n = 1$, $A_1 = A_1$.

   Inductive step: Use the fact that if $A \subseteq B$ then $A \cap B = A$. If $\bigcap_{i=1}^{n} A_i = A_n$, then

$$\begin{aligned}\bigcap_{i=1}^{n+1} A_i &= \bigcap_{i=1}^{n} A_i \cap A_{n+1} \\ &= A_n \cap A_{n+1} \\ &= A_{n+1}.\end{aligned}$$

2. What, if anything, is wrong with the following proof?

   Theorem 0.1 (Questionable Theorem) *Define f_n by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Then for all $n \geq 1$, $3f_n + f_{n+1} = 2 \cdot f_{n+2}$.*

   Proof: For the base case: $f_1 = 1, f_2 = 1$, and $f_3 = 2$, so when $n = 1$ we have $3f_n + f_{n+1} = 3f_1 + f_2 = 4 = 2 \cdot f_3 = 2 \cdot f_{n+2}$.

   Inductive step: Suppose that $3f_n + f_{n+1} = 2f_{n+2}$. Then

$$\begin{aligned}3f_{n+1} + f_{n+2} &= 3(f_n + f_{n-1}) + f_{n+1} + f_n \\ &= (3f_n + f_{n+1}) + (3f_{n-1} + f_n) \\ &= 2f_{n+2} + 2f_{n+1} \\ &= 2f_{n+3}.\end{aligned}$$

   By induction, $3f_n + f_{n+1} = 2f_{n+2}$ for all $n \geq 1$.

   Answer: The "proof" of the inductive step requires the formula to work for both n and $n-1$, not just n. We therefore need to check 2 base cases in a row for the proof to be valid. As it turns out, the theorem is incorrect.

3. Which Fibonacci numbers are even? Come up with a conjecture and prove it.

   If $3|n$ then f_n is even. Otherwise n is odd.

   Proof: Base case: $f_0 = 0$ is even while $f_1 = 1$ is odd.

   Inductive step: Suppose that the even-odd pattern holds for all $k \leq n$. Then if $3|n+1$, neither n nor $n-1$ are divisible by 3. Thus $f_{n+1} = f_n + f_{n-1}$ is the sum of two odd numbers and is therefore even.

   If $3 \nmid n+1$, exactly one of n and $n-1$ is divisible by 3. Thus $f_{n+1} = f_n + f_{n-1}$ is the sum of one even number and one odd number and so is odd.

   This completes the proof by induction— note that two consecutive examples were required for the base case.

Recursion

1. Define a sequence a_n by $a_0 = 1$, $a_1 = 3$ and $a_n = a_{n-1} + 2 \cdot a_{n-2}$ for $n \geq 2$. Find a_6. Prove that $a_n = \dfrac{2^{n+2} - (-1)^n}{3}$.

a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	3	5	11	21	43	85

Base case for proof by induction: The formula works for $n = 0$ and $n = 1$.

Inductive step: Suppose that the formula works for n AND $n + 1$. Then

$$\begin{aligned} a_{n+2} &= a_{n+1} + 2a_n \\ &= \frac{2^{n+3} - (-1)^{n+1}}{3} + 2 \cdot \frac{2^{n+2} - (-1)^n}{3} \\ &= \frac{2 \cdot 2^{n+3} - (-1)^n}{3} \\ &= \frac{2^{n+4} - (-1)^{n+2}}{3} \end{aligned}$$

Note that this time we needed to use the formula for both a_{n+1} and a_n, so we needed to prove two base cases.

2. Define a sequence a_n by $a_0 = 1$, $a_1 = 3$, and $a_n = \sum_{i=1}^{n-1} a_i$ for $n \geq 2$. Find a formula for a_n and prove that it is correct.

$a_2 = 4, a_3 = 8, a_4 = 16$. Conjecture that for $n \geq 2$, $a_n = 2^n$.

Base case: when $n = 2$, $a_2 = 4 = 2^2$.

Inductive step: If $a_n = 2^n$, then

$$\begin{aligned} a_{n+1} &= \sum_{i=1}^{n} a_i \\ &= \sum_{i=1}^{n-1} a_i + a_n \\ &= a_n + a_n \\ &= 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

This completes the proof by induction.

3. Prove: $\gcd(f_{n+1}, f_n) = 1$ for all $n \geq 0$.

Proof: $\gcd(f_0, f_1) = \gcd(0, 1) = 1$ for the base case $n = 0$.

Inductive step: use the fact that $\gcd(a, b) = \gcd(a - b, b)$. Then if the proposition holds for n, we have $\gcd(f_{n+2}, f_{n+1}) = \gcd(f_{n+2} - f_{n+1}, f_{n+1}) = \gcd(f_n, f_{n+1}) = 1$.

4. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for $n \geq 1$.

Base case: it works for $n = 1$ since $f_1^2 = 1 \cdot 1 = f_1 f_2$.

Inductive step: if the formula holds for n, then

$$\begin{aligned} (f_1^2 + f_2^2 + \cdots + f_n^2) + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1} f_{n+1} \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{n+2} \end{aligned}$$

5. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ for $n \geq 1$.

   Base case: $f_1 = 1 = f_2$ when $n = 1$.

   Inductive step: if the formula holds for n, then

$$\begin{aligned}(f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1} &= f_{2n} + f_{2n+1} \\ &= f_{2n+2}\end{aligned}$$

6. (★) Prove that $\gcd(f_m, f_n) = f_{\gcd(m,n)}$ for any $m, n \in \mathbb{N}$.

   Good luck!

Recursive Games

You and a friend are playing a game: there is a pile of stones. You take turns removing stones from the pile— during your turn, you may remove 1, 2, or 3 stones. Whoever removes the last stone wins.

1. Prove that the second player has a winning strategy if the pile begins with 8 stones.

   If player 1 takes n stones, then player 2 should take $(4 - n)$ stones, leaving 4 stones in the pile. Player 1 then has to take at least 1 stone, and player 2 can take the rest.

2. Who has a winning strategy if the pile begins with 15 stones? 16 stones?

   If the pile starts with 15 stones then player 1 can win by taking 3 stones to start. If the pile starts with 16 stones then player 2 can win.

3. Suppose that the pile has n stones, and you may choose whether to go first or second. How should you decide?

   If the pile begins with $4m$ stones for $n \geq 1$, player 2 has the winning strategy.

   Proof: For the base case $m = 1$ it is easy to see that player 2 can win.

   If player 2 can win a game with 4m stones and the game begins with $4(m + 1) = 4m + 4$ stones, then when player 1 takes k stones player 2 should take $(4 - k)$ stones, leaving the pile with $4m$. By the inductive hypothesis, player 2 can win this game.

   If the pile begins with $4m + 1$, $4m + 2$, or $4m + 3$ stones then player 1 can win by choosing 1, 2 or 3 stones (respectively) and leaving the pile with 4m. Player 1 can then use the same winning strategy that player 2 would have used in the previous scenario.