Chapter 5.1-5.3: More Induction and Recursion

Monday, October 5

Warmup

1. Prove: If $a_0 = 1$ and $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, then $a_n > 0$ for all $n \in \mathbb{N}$.
2. Define: $A(n) = \begin{cases} 1 & n = 0 \\ n \cdot A(n-1) & n \geq 1 \end{cases}$. What is $A(5)$? What is the function A?
3. Define: $B(n) = \begin{cases} 0 & n = 0 \\ n + B(n-1) & n \geq 1 \end{cases}$. What is $B(5)$? What is the function B?
4. Define: $C(x, y) = \begin{cases} x & y = 0 \\ C(y, x) & y > x \\ C(x - y, y) & x \geq y > 0 \end{cases}$. What is $C(22, 6)$? What is the function C?

Recursive Structures

Find ways to define the following expressions recursively over the variable n (for $n \in \mathbb{N}$):

1. $\sum_{i=1}^{n} a_i$
2. x^n
3. $n!$
4. $\bigcup_{i=1}^{n} A_i$
5. The song "n Bottles of Beer on the Wall"
6. $\max(a_1, a_2, \ldots, a_n)$
7. A function that takes a finite list of integers and returns 1 if all of the integers are positive and 0 otherwise.
8. A function that tells you whether a given word is a palindrome.

From Two to Many

1. Given that $ab = ba$, prove that $a^n b = ba^n$ for all $n \geq 1$.

2. Given: if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$. Prove: if $a_i \equiv b_i \pmod{m}$ for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} a_i \equiv \sum_{i=1}^{n} b_i \pmod{m}$.

3. Prove: $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$.

4. Given: $(fg)' = f'g + fg'$. Prove: $(fgh)' = f'gh + fg'h + fgh'$. Also prove: $\left(\prod_{i=1}^{n} f_i\right)' = \sum_{i=1}^{n} f_i' \cdot \prod_{j \neq i} f_j$.

5. Given: if A and B are countable then $A \times B$ is countable. Prove: $\mathbb{Z}^n$ is countable for any $n \geq 1$.