Chapter 5.1-5.3: More Induction and Recursion

Monday, October 5

Warmup

1. Prove: If $a_0 = 1$ and $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, then $a_n > 0$ for all $n \in \mathbb{N}$.

   Base case: $a_0 > 0$. Check!

   Inductive step: If $a_n > 0$ then $an + 1 \geq a_n > 0$, so $a_{n+1} > 0$. All done!

2. Define: $A(n) = \begin{cases} 1 & n = 0 \\ n \cdot A(n-1) & n \geq 1 \end{cases}$. What is $A(5)$? What is the function A?

   $A(5) = 120$. $A(n) = n!$.

3. Define: $B(n) = \begin{cases} 0 & n = 0 \\ n + B(n-1) & n \geq 1 \end{cases}$. What is $B(5)$? What is the function B?

   $B(5) = 15$. $B(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

4. Define: $C(x, y) = \begin{cases} x & y = 0 \\ C(y, x) & y > x \\ C(x - y, y) & x \geq y > 0 \end{cases}$. What is $C(22, 6)$? What is the function C?

   $C(22, 6) = C(16, 6) = C(10, 6) = C(4, 6) = C(6, 4) = C(2, 4) = C(4, 2) = C(0, 2) = C(2, 0) = 2$. C is the greatest common divisor function.

Recursive Structures

Find ways to define the following expressions recursively over the variable n (for $n \in \mathbb{N}$):

1. $\sum_{i=1}^{n} a_i = a_n + \sum_{i=1}^{n-1} a_i$, and simply a_1 if $n = 1$.
2. x^n: 1 if $n = 0$, $x \cdot x^{n-1}$ otherwise.
3. $n!$: 0 if $n = 0$, $n \cdot (n-1)!$ otherwise.
4. $\bigcup_{i=1}^{n} A_i$: A_1 if $n = 1$, $A_n \cup \bigcup_{i=1}^{n-1} A_i$ otherwise.
5. The song "n Bottles of Beer on the Wall": nothing if $n = 0$, one verse followed by "$(n-1)$ Bottles of Beer on the Wall" otherwise.
6. $\max(a_1, a_2, \ldots, a_n) = \max(a_n, \max(a_1, \ldots, a_{n-1}))$ if $n > 2$, and $\max(a_1, a_2)$ if $n = 2$.
7. A function that takes a finite list of integers and returns 1 if all of the integers are positive and 0 otherwise.

   $f(a_1, \ldots, a_n) = \begin{cases} 0 & a_n \leq 0 \\ 1 & \text{List is empty} \\ f(a_1, \ldots, a_{n-1}) & a_n > 0 \end{cases}$.

8. A function that tells you whether a given word is a palindrome.

   $f(\text{"}abc \ldots c'b'a'\text{"}) = \mathbf{T}$ if the word has 0 letters or 1 letter, $\mathbf{F}$ if $a \neq a'$, and $f(\text{"}bc \ldots c'b'\text{"})$ if $a = a'$.

From Two to Many

1. Given that $ab = ba$, prove that $a^n b = ba^n$ for all $n \geq 1$.

   Base case: $ab = ba$ is given.

   Inductive step: If $a^{n-1}b = ba^{n-1}$ then $a^n b = a \cdot a^{n-1}b = a \cdot ba^{n-1} = b \cdot a \cdot a^{n-1} = ba^n$. Done.

2. Given: if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$. Prove: if $a_i \equiv b_i \pmod{m}$ for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^n a_i \equiv \sum_{i=1}^n b_i \pmod{m}$.

   Base case: When $n = 2$ the formula $a + c \equiv b + d \pmod{m}$ was already given.

   Inductive step: Supposing the formula works for n, we get

$$\begin{aligned}\sum_{i=1}^{n+1} a_i &= (\sum_{i=1}^{n} a_i) + a_{n+1} \\ &\equiv \sum_{i=1}^{n} b_i + b_{n+1} \\ &\equiv \sum_{i=1}^{n+1} b_i\end{aligned}$$

3. Prove: $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$.

   Base case: When $n = 2$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$ is given by one of DeMorgan's Laws.

   Inductive step: Suppose the formula works for n. Then

$$\begin{aligned}\overline{\bigcup_{i=1}^{n+1} A_i} &= \overline{\bigcup_{i=1}^{n} A_i \cup A_{n+1}} \\ &= \overline{\bigcup_{i=1}^{n} A_i} \cap \overline{A_{n+1}} \\ &= \bigcap_{i=1}^{n} \overline{A_i} \cap \overline{A_{n+1}} \\ &= \bigcap_{i=1}^{n+1} \overline{A_i}\end{aligned}$$

4. Given: $(fg)' = f'g + fg'$. Prove: $(fgh)' = f'gh + fg'h + fgh'$. Also prove: $(\prod_{i=1}^n f_i)' = \sum_{i=1}^n f_i' \cdot \prod_{j \neq i} f_j$.

   Base case: when $n = 2$ the formula is given.

Inductive step: Suppose the formula works for n. Then

$$\begin{aligned}
\left(\prod_{i=1}^{n+1} f_i\right)' &= \left(\prod_{i=1}^{n} f_i \cdot f_{n+1}\right)' \\
&= \left(\prod_{i=1}^{n+1} f_i\right)' f_{n+1} + \left(\prod_{i=1}^{n+1} f_i\right) f'_{n+1} \\
&= \sum_{i=1}^{n} f'_i \cdot \prod_{j\neq i} f_j + f'_{n+1} \prod_{j=1}^{n} f_j \\
&= \sum_{i=1}^{n+1} f'_i \cdot \prod_{j\neq i} f_j
\end{aligned}$$

5. Given: if A and B are countable then $A \times B$ is countable. Prove: $\mathbb{Z}^n$ is countable for any $n \geq 1$.

   Base case: When $n = 1$ we know that $\mathbb{Z}$ is countable. When $n = 2$ we know that $\mathbb{Z} \times \mathbb{Z}$ is countable.

   Inductive step: If $\mathbb{Z}^{n-1}$ is countable then $\mathbb{Z}^n = \mathbb{Z}^{n-1} \times \mathbb{Z}$, which is countable because it is the product of two countable sets.