Chapter 5.1: Induction and Recursion

Wednesday, September 30

Warmup

Define: A *curious* number is a number that is curious. Suppose we know two things about curious numbers:

1. If any integer n is a curious number, then $n + 2$ is a curious number.
2. 7 is a curious number.

Which numbers *must* also be curious?

1. 5
2. 9
3. 10
4. 15
5. 1341
6. 2808
7. 39523092357
8. n
9. ∞

All odd numbers greater than 7 must be curious. No others are necessarily curious.

Now suppose that 10 is *not* a curious number. Which number must *not* be curious?
No even numbers less than or equal to 10 can be curious.

There is a machine that makes widgets all day. It has one problem— if a widget it makes is defective, then the next widget it makes will also be defective. What can you say about the machine?
If the machine ever makes a defective widget, it will only make defective widgets from that point on.

Induction

1. For what integers is $2^n \geq n^2$ true? Prove it.

   $2^n \geq n^2$ for $n = 0, 1, 2,$ and $n \geq 4$.

   Base case: $2^4 = 16 = 4^2$.

   Inductive step. Suppose that $n \geq 4$ and $2^n \geq n^2$. Then

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\geq n^2 + n^2 \\ &\geq n^2 + 4n \\ &\geq n^2 + 2n + 2n \\ &\geq n^2 + 2n + 1 \\ &= (n+1)^2. \end{aligned}$$

2. (Calculus) Suppose we know that $\frac{d}{dx}x = 1$ and that for any functions f and g, $(fg)' = f'g + fg'$. Prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \geq 1$.

   Base case: when $n = 1$, $\frac{d}{dx}x^1 = 1 = 1 \cdot x^0$.

   Inductive step: If $\frac{d}{dx}x^n = nx^{n-1}$, then

$$\begin{aligned} \frac{d}{dx}x^{n+1} &= (x \cdot x^n)' \\ &= x' \cdot x^n + (x^n)' \cdot x \\ &= x^n + nx^{n-1} \cdot x \\ &= (n+1)x^n \end{aligned}$$

3. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for $n \geq 0$.

   Base case: it works for $n = 0$ since $0 = 0(0+1)(0+2)/6$.

   Inductive step. Suppose that the formula works for n. Then

$$\begin{aligned} (1^2 + 2^2 + \cdots + n^2) + (n+1)^2 &= n(n+1)(2n+1)/6 + n^2 + 2n + 1 \\ &= \frac{2n^3 + 3n^2 + 2n + 6n^2 + 12n + 6}{6} \\ &= \frac{2n^3 + 9n^2 + 14n + 6}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

4. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for $n \geq 1$.

   Base case: it works for $n = 1$.

   Inductive step: suppose it works for n. Then

$$\begin{aligned} (1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!) + (n+1) \cdot (n+1)! &= (n+1)! - 1 + [(n+2) \cdot (n+1)! - (n+1)!] \\ &= (n+2)! - 1 \end{aligned}$$

5. Find a closed form for $\sum_{k=1}^{n}(-1)^k k^2$ and prove that it is correct.

The first few terms are $-1, 3, -6, 10, -15, \ldots$, so guess that the formula is $(-1)^n n(n+1)/2$.

Base case: The formula works for $n = 1$.

Inductive step: suppose that it works for n. Then

$$\begin{aligned}
\sum_{k=1}^{n+1}(-1)^k k^2 &= \sum_{k=1}^{n}(-1)^k k^2 + (-1)^{n+1}(n+1)^2 \\
&= (-1)^n n(n+1)/2 + (-1)^{n+1}(n^2+2n+1) \\
&= (-1)^{n+1}\frac{2n^2+4n+2-n^2-n}{2} \\
&= (-1)^{n+1}\frac{n^2+3n+2}{2} \\
&= (-1)^{n+1}\frac{(n+1)(n+2)}{2}
\end{aligned}$$