Necessary and Sufficient Conditions

Monday, September 28

Key Topics

- p is *sufficient* for q if $p \Rightarrow q$.
- p is *necessary* for q if $\neg p \Rightarrow \neg q$ (or, $q \Rightarrow p$).
- p is *necessary and sufficient* for q if $p \Leftrightarrow q$.

Warmup

Theorem 0.1 (Primality Test) *If p is a prime number and $p > 2$ then $2^{p-1} \equiv 1 \pmod{p}$.*

1. If $2^{128} \equiv 4 \pmod{129}$, what can you conclude? **129 is not prime.**
2. If $2^{560} \equiv 1 \pmod{561}$, what can you conclude? **Nothing.**
3. If n is a number such that $2^{n-1} \equiv 0 \pmod{n}$, what can you conclude? **Either $n \leq 2$ or n is not prime.**
4. How can you tell for certain that a number is prime? **The simplest way is to try dividing n by every prime number less than its square root. There are more efficient tests, but they are beyond the scope of this course.**

Theorem 0.2 (Raven Theorem) *All ravens are black.*

Suppose we want to find a counterexample to the statement "All birds are black." What information does Theorem 0.2 give us about such a counterexample? **The counterexample must be a bird that is not a raven.**

Let x be a real number, and say we want to ensure that $x^2 + 1 > 5$. Find conditions on x that are...

1. Necessary and sufficient. $|x| > 2$
2. Necessary, but not sufficient. $x \neq 0$
3. Sufficient, but not necessary. $x > 10$
4. Neither necessary nor sufficient. $x > 0$

Hypothesis Testing

Theorem 0.3 (Bezout's Theorem) *If* $\gcd(a, b) = 1$ *then there exist* x *and* y *such that* $ax + by = 1$.

Theorem 0.4 (The Prime Property) *If* p *is prime, then* p *has the following property: if* $p|ab$ *then* $p|a$ *or* $p|b$.

1. Suppose we have two integers a and b and want to find x and y such that $ax + by = 1$. We know that the condition $\gcd(a, b) = 1$ is sufficient, but is it necessary?

   Yes: suppose that $\gcd(a, b) = d > 1$**, then** $ax + by = dnx + dmy = d(nx + my)$**, which is divisible by** d **and therefore not equal to 1.**

2. Suppose we want to find x and y such that $ax + by = 7$. Is the condition $\gcd(a, b) = 1$ necessary? Is it sufficient?

   It is sufficient, since if $ax+by = 1$ **then** $(7x)a+(7y)b = 7$**. It is not necessary. . . take** $a = 7, b = 0$ **as a counterexample.**

3. Consider the statement "If $\gcd(a, b) \leq 3$ then there exist x and y such that $ax + by = 1$." What must a counterexample to this statement look like?

   Must have $\gcd(a, b) = 2$ **or** 3**.**

4. Is it necessary that p be prime in order for it to have the prime property?

   No. p **can also be** 0 **or** 1**.**

5. Prove or find a counterexample: If p is prime and $ab \equiv 0 \pmod{p}$ then $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$.

   True, since when put in divisibility notation this is the same as the Prime property.

6. Prove or find a counterexample: If $n \geq 2$ and $ab \equiv 0 \pmod{n}$ then $a \equiv 0 \pmod{n}$ or $b \equiv 0 \pmod{n}$.

   False. $n = 6, a = 3, b = 2$ **is a counterexample.**