Chapters 2.2-2.5: Sets and Functions

Monday, September 14

Key Notes

- One way to prove $A = B$: prove $A \subset B$ and $B \subset A$.
- $f : A \to B$ is 1-1 if $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.
- $f : A \to B$ is 1-1 if $a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$.
- $f : A \to B$ is onto if $(\forall b \in B)(\exists a \in A)(f(a) = b)$.

Warmup

1. What rule in propositional logic does $\overline{A \cup B} = \overline{A} \cap \overline{B}$ correspond to?

   De Morgan's Law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

2. How can you say "f is not onto" in quantifier notation? How do you prove that a function is not onto?

   $(\exists b \in B)(\forall a \in A)(f(a) \neq b)$, or $(\exists b \in B)(\neg\exists a \in A)(f(a) = b)$. Find an element of b that f does not map any element to.

Set Proofs 1

Consider the statement "If $A \subseteq B$ then $A \cup C \subseteq B \cup C$."

1. Find an analogous statement in propositional logic. $(p \Rightarrow q) \Rightarrow ((p \vee r) \Rightarrow (q \vee r))$.

2. Draw a Venn Diagram that illustrates the proof.

3. Make a given-goal diagram for the proof.

Given	Goal
$A \subseteq B$	$A \cup C \subseteq B \cup C$

   Can be strengthened to the following:

Given	Goal
$A \subseteq B$	$x \in B$ or $x \in C$
$x \in A \cup C$	

   This can be strengthed again, if you like:

Given	Goal
$A \subseteq B$	$x \in B$
$x \in A \cup C$	
$x \notin C$	

4. Prove the statement. Prove or disprove the converse.

   Suppose $x \in A \cup C$. If $x \in C$, then $x \in B \cup C$. If $x \in A$, then $x \in B$ because $A \subseteq B$, and so $x \in B \cup C$. Therefore, if $A \subset B$ then $A \cup C \subseteq B \cup C$.

Do the same with the statement "If $A \subset B$ then $B \cup \overline{A} = U$."
Proposition: $(p \Rightarrow q) \Rightarrow (q \vee \neg p) \equiv \mathbf{T}$.
To prove: we know that $B \cup \overline{A} \subset U$ is always true, so we just have to prove that if $x \in U$ then $x \subset B \cup \overline{A}$.

Given	Goal
$A \subset B$	$x \in B$ or $x \in \overline{A}$.

This can be rearranged to get the following:

Given	Goal
$A \subset B$	$x \in B$
$x \notin \overline{A}$	

Proof: We want to show that $x \in \overline{A} \cup B$ for any x. If $x \in \overline{A}$ then we are done. Otherwise, $x \in A$, which means that $x \in B$ because $A \subset B$. Therefore $B \cup \overline{A} = U$.

Set Proofs 2

Prove the following statements using a series of equivalences.

1. $A - (A - C) \subseteq C$

$$\begin{aligned} A - (A - C) &= A \cap \overline{A \cap \overline{C}} \\ &= A \cap (\overline{A} \cup \overline{C}) \\ &= (A \cap \overline{A}) \cup (A \cap \overline{\overline{C}}) \\ &= \emptyset \cup (A \cap C) \\ &= A \cap C \\ &\subset C \end{aligned}$$

2. $\overline{A \cup B} \cap B = \emptyset$

$$\begin{aligned} \overline{A \cup B} \cap B &= \overline{A} \cap \overline{B} \cap B \\ &= \overline{A} \cap \emptyset \\ &= \emptyset. \end{aligned}$$

3. $B = (A \cap B) \cup (\overline{A} \cap B)$

$$\begin{aligned} (A \cap B) \cup (\overline{A} \cap B) &= (A \cup \overline{A}) \cap B \\ &= U \cap B \\ &= B \end{aligned}$$

Functions

1. Prove that $f(x) = 3x$ is onto if f is from $\mathbb{R}$ to $\mathbb{R}$ but not if f is from $\mathbb{Z}$ to $\mathbb{Z}$.

   In the first case: for any b, $f(b/3) = b$.

   This does not work for $\mathbb{Z}$ because (for example) $1/3$ is not an integer, so there is no x such that $f(x) = 1$.

2. Prove that $f(x) = 3x$ is 1-1 in either of the above cases.

   If $f(a) = f(b)$ then $3a = 3b$, so $a = b$.

3. Say that a function is *increasing* if $a > b$ implies that $f(a) > f(b)$ for all a, b in the domain of f. Prove that increasing functions are 1-1.

   Supposing that $a \neq b$, we want to prove that $f(a) \neq f(b)$. If $a \neq b$, then $a > b$ or $a < b$. In the first case we get $f(a) > f(b)$ and in the second we get $f(a) < f(b)$. Either way, $f(a) \neq f(b)$.

4. Prove that the sum of two increasing functions is increasing.

   Take any $a > b$ and increasing functions f and g. Then $(f + g)(a) = f(a) + g(a) > f(b) + g(a) > f(b) + g(b) = (f + g)(b)$, so $f + g$ is increasing.

5. Prove or disprove: the sum of two 1-1 functions is 1-1.

   False: $f(x) = x$ and $g(x) = -x$ are both 1-1 but the sum is $(f + g)(x) = 0$, which is not one-to one on any domain with more than one element.

Cardinality

1. Prove that if A and B are countable then $A \cup B$ is countable.

   Informal explanation: $|A \cup B| \leq |A| + |B|$, and since A and B are countable we can count the element of A and B together by alternating between the two sets.

   Formal proof: there is a bijection f between A and the even numbers and a bijection g between B and the odd numbers since A and B are countable. Then define $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \notin A \end{cases}$.

   h is therefore an injection from $A \cup B$ to $\mathbb{N}$. Note that we had to use the condition "$x \notin A$" rather than "$x \in B$" because it is possible for some element x to be in both A and B, in which case the function would not be well-defined for x.

2. Prove that if A is countable and B is uncountable then $B - A$ is uncountable.

   Suppose that A is countable and $B - A$ is countable, then by the previous problem B would be countable. By the contrapositive, if A is countable but B is uncountable, then $B - A$ must be uncountable. A proof by contradiction would also work here.

   As an example: $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, so the set of irrational numbers $\mathbb{R} - \mathbb{Q}$ is uncountable.