Chapters 1.7-2.2: Proofs, some Sets

Wednesday, September 9

Key Notes

- Direct proof: to prove $A \to B$, take A as a given and prove B.
- Proof by contraposition: to prove $A \to B$, assume $\neg B$ and prove $\neg A$.
- To prove $A \Leftrightarrow B$, prove the forward/contrapositive *and* inverse/converse.
- All statements about sets have an analogous form in predicate logic.

Warmup

1. Come up with English sentences to illustrate each of the following tautologies:
   (a) $((p \to q) \land p) \to q$
   (b) $((p \to q) \land \neg q) \to \neg p$
   (c) $((p \to q) \land (q \to r)) \to (p \to r)$
   (d) $((p \lor q) \land \neg p) \to q$
   (e) $((p \to q) \land (\neg p \to r)) \to (q \lor r)$
2. What, if anything, is wrong with the following reasoning?

   "Suppose that x is a number such that $\sqrt{2x^2 - 1} = x$. Then

$$\begin{aligned} \sqrt{2x^2-1} &= x \\ 2x^2 - 1 &= x^2 \\ x^2 - 1 &= 0 \\ (x+1)(x-1) &= 0 \\ x &= \pm 1 \end{aligned}$$

   Therefore, if $\sqrt{2x^2 - 1} = x$ then $x = \pm 1$."

Biconditionals

State the inverse, converse, and contrapositive of each of these statements. Which forms apppear to be the simplest to prove? Prove the statements.

1. $x = 3$ if and only if $x^2 = 9$ and $x \neq -3$.
2. $x^2 = 1$ if and only if $x = 1$ or $x = -1$ (one direction requires a proof by cases).
3. $\max(a, b) = \min(a, b)$ if and only if $a = b$.

Backward Reasoning

For which of these claims is the converse true? If false, give a counterexample.

1. If $a = b$ then $ac = bc$.
2. If $a = b$ then $a^2 = b^2$.
3. Suppose $c \neq 0$. If $a = b$ then $ac = bc$.
4. If $a > 0$ then $a^2 > 0$.
5. If $a > 0$ then $1/a > 0$.
6. Suppose $a, b \geq 0$. If $a > b$ then $a^2 > b^2$.

Backwards Proofs

1. Prove that $(x-3)^2 + (x+3)^2 = 2(x+3)(x-3) + 36$.
2. Prove that if $x, y > 0$ then $\frac{2xy}{x+y} \leq \frac{x+y}{2}$, with equality if and only if $x = y$.

Set Operations

What can you say about the sets A and B if

1. $A \cup B = A$?
2. $A - B = A$?
3. $A - B = B - A$?
4. $A \cup B = B \cup A$?

Relation to Propositional Logic

1. What rule in propositional logic does the identity $A \cap U = A$ correspond to? What does U correspond to?
2. What does $\overline{A \cup B} = \overline{A} \cap \overline{B}$ correspond to?
3. What does $\emptyset$ correspond to?
4. Prove: if $A \subset B$ and $B \subset C$ then $A \subset C$. What rule of inference does this correspond to?
5. Prove: if $A \subset B$ and $\overline{A} \subset C$ then $B \cup C = U$. What rule of inference does this correspond to?