Key Notes

- Direct proof: to prove $A \rightarrow B$, take $A$ as a given and prove $B$.
- Proof by contraposition: to prove $A \rightarrow B$, assume $\neg B$ and prove $\neg A$.
- To prove $A \iff B$, prove the forward/contrapositive and inverse/converse.
- All statements about sets have an analogous form in predicate logic.

Warmup

1. Come up with English sentences to illustrate each of the following tautologies:
   
   (a) $((p \rightarrow q) \land p) \rightarrow q$
   
   (b) $((p \rightarrow q) \land \neg q) \rightarrow \neg p$
   
   (c) $((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$
   
   (d) $((p \lor q) \land \neg p) \rightarrow q$
   
   (e) $((p \rightarrow q) \land (\neg p \rightarrow r)) \rightarrow (q \lor r)$

2. What, if anything, is wrong with the following reasoning?

   “Suppose that $x$ is a number such that $\sqrt{2x^2 - 1} = x$. Then
   
   $\sqrt{2x^2 - 1} = x$
   
   $2x^2 - 1 = x^2$
   
   $x^2 - 1 = 0$
   
   $(x + 1)(x - 1) = 0$
   
   $x = \pm 1$

   Therefore, if $\sqrt{2x^2 - 1} = x$ then $x = \pm 1$.”

Biconditionals

State the inverse, converse, and contrapositive of each of these statements. Which forms appear to be the simplest to prove? Prove the statements.

1. $x = 3$ if and only if $x^2 = 9$ and $x \neq -3$.

2. $x^2 = 1$ if and only if $x = 1$ or $x = -1$ (one direction requires a proof by cases).

3. $\max(a, b) = \min(a, b)$ if and only if $a = b$. 
Backward Reasoning

For which of these claims is the converse true? If false, give a counterexample.

1. If \(a = b\) then \(ac = bc\).
2. If \(a = b\) then \(a^2 = b^2\).
3. Suppose \(c \neq 0\). If \(a = b\) then \(ac = bc\).
4. If \(a > 0\) then \(a^2 > 0\).
5. If \(a > 0\) then \(1/a > 0\).
6. Suppose \(a, b \geq 0\). If \(a > b\) then \(a^2 > b^2\).

Backwards Proofs

1. Prove that \((x - 3)^2 + (x + 3)^2 = 2(x + 3)(x - 3) + 36\).
2. Prove that if \(x, y > 0\) then \(\frac{2xy}{x+y} \leq \frac{x+y}{2}\), with equality if and only if \(x = y\).

Set Operations

What can you say about the sets \(A\) and \(B\) if

1. \(A \cup B = A?\)
2. \(A - B = A?\)
3. \(A - B = B - A?\)
4. \(A \cup B = B \cup A?\)

Relation to Propositional Logic

1. What rule in propositional logic does the identity \(A \cap U = A\) correspond to? What does \(U\) correspond to?
2. What does \(\overline{A \cup B} = \overline{A} \cap \overline{B}\) correspond to?
3. What does \(\emptyset\) correspond to?
4. Prove: if \(A \subset B\) and \(B \subset C\) then \(A \subset C\). What rule of inference does this correspond to?
5. Prove: if \(A \subset B\) and \(\overline{A} \subset C\) then \(B \cup C = U\). What rule of inference does this correspond to?