Chapters 1.7-2.2: Proofs, some Sets — Solutions

Wednesday, September 9

Key Notes

- Direct proof: to prove $A \to B$, take A as a given and prove B.
- Proof by contraposition: to prove $A \to B$, assume $\neg B$ and prove $\neg A$.
- To prove $A \Leftrightarrow B$, prove the forward/contrapositive *and* inverse/converse.
- All statements about sets have an analogous form in predicate logic.

Warmup

1. Come up with English sentences to illustrate each of the following tautologies:

   (a) $((p \to q) \land p) \to q$
   If it is raining then I am happy. It is raining. Therefore, I am happy.

   (b) $((p \to q) \land \neg q) \to \neg p$
   If it is raining then I am happy. I am not happy. Therefore, it is not raining.

   (c) $((p \to q) \land (q \to r)) \to (p \to r)$
   If it rains then I will be happy. If I am happy then I will bake a cake. Therefore, if it rains then I will bake a cake.

   (d) $((p \lor q) \land \neg p) \to q$
   Either the Cubs or the Sox won yesterday. The Cubs did not win yesterday. Therefore, the Sox won yesterday.

   (e) $((p \to q) \land (\neg p \to r)) \to (q \lor r)$
   If it rains tomorrow I will stay inside. If it does not rain tomorrow I will have a picnic. Therefore, I will either stay inside or have a picnic tomorrow.

2. What, if anything, is wrong with the following reasoning?

   "Suppose that x is a number such that $\sqrt{2x^2-1} = x$. Then

$$\begin{aligned} \sqrt{2x^2-1} &= x \\ 2x^2-1 &= x^2 \\ x^2-1 &= 0 \\ (x+1)(x-1) &= 0 \\ x &= \pm 1 \end{aligned}$$

   Therefore, if $\sqrt{2x^2-1} = x$ then $x = \pm 1$."

   Nothing…the statements is true! But the converse, "If $x = \pm 1$ then $\sqrt{2x^2-1} = x$,", is false since $x = -1$ is not a solution.

Biconditionals

State the inverse, converse, and contrapositive of each of these statements. Which forms apppear to be the simplest to prove? Prove the statements.

1. $x = 3$ if and only if $x^2 = 9$ and $x \neq -3$.

   Forward: If $x = 3$ then $x^2 = 9$ and $x \neq -3$.

   Inverse: If $x \neq 3$ then $x^2 \neq 9$ or $x = -3$.

   Converse: If $x^2 = 9$ and $x \neq -3$, then $x = 3$.

   Contrapositive: If $x^2 \neq 9$ or $x = -3$, then $x \neq 3$.

   The forward and converse are easiest to prove. The forward should be very simple. For the converse, if $x^2 = 9$ then $x = \pm 3$. Since $x \neq -3$, it follows that $x = 3$.

2. $x^2 = 1$ if and only if $x = 1$ or $x = -1$ (one direction requires a proof by cases).

   Forward: If $x^2 = 1$ then $x = 1$ or $x = -1$.

   Inverse: If $x^2 \neq 1$ then $x \neq 1$ and $x \neq -1$.

   Converse: If $x = 1$ or $x = -1$ then $x^2 = 1$.

   Contrapositive: If $x \neq 1$ and $x \neq -1$ then $x^2 \neq 1$.

   The forward and converse are easiest to prove. For the forward, we can say $0 = x^2 - 1 = (x+1)(x-1)$ and conclude that $x + 1 = 0$ or $x - 1 = 0$.

   The converse is a small proof by cases: If $x = 1$ then $x^2 = 1$. If $x = -1$ then $x^2 = 1$. Therefore if $x = \pm 1$ then $x^2 = 1$.

3. $\max(a, b) = \min(a, b)$ if and only if $a = b$.

   Forward: If $\max(a, b) = \min(a, b)$ then $a = b$.

   Inverse: If $\max(a, b) \neq \min(a, b)$ then $a \neq b$.

   Converse: If $a = b$ then $\max(a, b) = \min(a, b)$.

   Contrapositive: If $a \neq b$ then $\max(a, b) \neq \min(a, b)$.

   The converse is easiest: if $a = b$ then $\max(a, b) = a = b = \min(a, b)$.

   One way to prove the contrapositive uses a proof by cases: $If a \neq b$ then $a > b$ or $a < b$. If $a > b$ then $\max(a, b) = a > b = \min(a, b)$. If $a < b$ then $\max(a, b) = b > a = \min(a, b)$.

Backward Reasoning

For which of these claims is the converse true? If false, give a counterexample.

1. If $a = b$ then $ac = bc$. The converse is false: take $a = 1, b = 2, c = 0$.
2. If $a = b$ then $a^2 = b^2$. The converse is false: take $a = -1, b = 1$.
3. Suppose $c \neq 0$. If $a = b$ then $ac = bc$. The converse is true, since we can divide both sides of $ac = bc$ by c.
4. If $a > 0$ then $a^2 > 0$. The converse is false: $a = -1$.
5. If $a > 0$ then $1/a > 0$. The converse is true.
6. Suppose $a, b \geq 0$. If $a > b$ then $a^2 > b^2$. The converse is true.

Backwards Proofs

1. Prove that $(x-3)^2 + (x+3)^2 = 2(x+3)(x-3) + 36$.

$$\begin{aligned}(x-3)^2 + (x+3)^2 = 2(x+3)(x-3) + 36 &\Leftrightarrow x^2 - 6x + 9 + x^2 + 6x + 9 = 2(x^2 - 9) + 36 \\ &\Leftrightarrow 2x^2 + 18 = 2x^2 - 18 + 36 \\ &\Leftrightarrow 2x^2 + 18 = 2x^2 + 18\end{aligned}$$

The last statement is true, so (because our statements were connect by "if and only iff" conditions) we can conclude that the initial proposition is true.

We could also rewrite this proof as one long string of equalities. This has the advantage of being easier to read, but it would be much harder to prove the equalities in this order:

$$\begin{aligned}(x-3)^2 + (x+3)^2 &= 2x^2 + 18 \\ &= 2x^2 - 18 + 36 \\ &= 2(x^2 - 9) + 36 \\ &= 2(x+3)(x-3) + 36\end{aligned}$$

2. Prove that if $x, y > 0$ then $\frac{2xy}{x+y} \leq \frac{x+y}{2}$, with equality if and only if $x = y$.

$$\begin{aligned}&\frac{2xy}{x+y} \leq \frac{x+y}{2} \\ &\Leftrightarrow 4xy \leq (x+y)^2 \\ &\Leftrightarrow 4xy \leq x^2 + 2xy + y^2 \\ &\Leftrightarrow 0 \leq x^2 - 2xy - y^2 \\ &\Leftrightarrow 0 \leq (x-y)^2\end{aligned}$$

The last statement is always true. Since the propositions are connected by biconditionals, we conclude that if $x, y > 0$ then $\frac{2xy}{x+y} \leq \frac{x+y}{2}$. The condition that $x, y > 0$ is used to go from the first inequality to the second: since $(x+y) > 0$, multiplying both sides by $(x+y)$ does not change the direction of the inequality.

Set Operations

What can you say about the sets A and B if

1. $A \cup B = A$? $B \subset A$.
2. $A - B = A$? $B \subset \overline{A}$.
3. $A - B = B - A$? $A = B$.
4. $A \cup B = B \cup A$? Nothing...this is true for any two sets A and B.

Relation to Propositional Logic

1. What rule in propositional logic does the identity $A \cap U = A$ correspond to? What does U correspond to?

   This identity corresponds to the rule $p \wedge \mathbf{T} \equiv p$. U (the universal set) corresponds to $\mathbf{T}$.

2. What does $\overline{A \cup B} = \overline{A} \cap \overline{B}$ correspond to?

   De Morgan's Law.

3. What does $\emptyset$ correspond to?

   $\mathbf{F}$.

4. Prove: if $A \subset B$ and $B \subset C$ then $A \subset C$. What rule of inference does this correspond to?

   Suppose $x \in A$. Then $x \in B$ (since $A \subset B$) and so $x \in C$ (since $B \subset C$). Therefore $A \subset C$.

   This corresponds to the rule Hypothetical Syllogism (if $p \Rightarrow q$ and $q \Rightarrow r$ then $p \Rightarrow r$).

5. Prove: if $A \subset B$ and $\overline{A} \subset C$ then $B \cup C = U$. What rule of inference does this correspond to?

   Take some $x \in U$. We consider two cases: either $x \in A$ or $x \notin A$ (in which case $x \in \overline{A}$).

   If $x \in A$ then $x \in B$ and so $x \in B \cup C$.

   If $x \in \overline{A}$ then $x \in C$ and so $x \in B \cup C$.

   Either way, $x \in B \cup C$ for any $x \in U$ and so $B \cup C = U$. This corresponds to (a variation of) the rule of Resolution: if $p \Rightarrow q$ and $\neg p \Rightarrow r$ then $q \vee r$ is always true.