Chapter 10.7: Planar Graphs

Wednesday, December 2

Summary

- A *planar graph* can be drawn in the plane so that no edges intersect. Such a drawing is a *plane graph*.
- Euler's Formula: For a plane graph, $v - e + r = 2$.
- If $v \geq 3$ then $e \leq 3v - 6$.
- Every planar graph has a vertex of degree ≤ 5.
- If G is triangle-free and $v \geq 3$ then $e \leq 2v - 4$
- Kuratowski's Theorem: a graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Euler's Formula

1. A connected plane graph has six vertices, each of degree four. Into how many regions does the graph divide the plane?

   The graph has $(6 \cdot 4)/2 = 12$ edges total, so using the formula $v - e + r = 2$ gives $6 - 12 + r = 2$, so $r = 8$. One example of such a graph is the plane grpah of the octahedron.

2. Find all five Platonic solids. Each one can be described by gluing regular $n-$gons together with m polygons to a vertex. For example, a cube is made of squares with 3 squares to a vertex and has 6 faces. Find the number of faces of the other Platonic solids.

   (a) Tetrahedron: 4 triangles, 3 to a vertex.
   (b) Octahedron: 8 triangles, 4 to a vertex.
   (c) Icosahedron: 20 triangles, 5 to a vertex.
   (d) Cube: 6 squares, 3 to a vertex.
   (e) Dodecahedron: 12 pentagons, 3 to a vertex.

3. If you have a solid block made of triangles and octagons with 3 shapes to a vertex, what can you say about the relation between the number of triangles and the number of octagons? (One of these shapes is made by slicing the vertices off of a cube.)

   We know that $v - e + r = 2$ for the tiling. Each triangle counts as $3/3$ vertices, 1 face, and $3/2$ edges, and so adds $1/2$ to the total. Each octagon counts as $8/3$ vertices, 1 face, and $8/2$ edges, and so subtracts $1/3$ from the total. Thus $\frac{1}{2}T - \frac{1}{3}O = 2$, or $3T - 2O = 12$. One option is 4 triangles and 0 octagons (a tetrahedron), and cutting the edges off of a cube gives 8 triangles and 6 octagons.

4. What if you have a solid block made of triangles and squares with 4 shapes to a vertex?

   We know that $v - e + r = 2$ for the tiling. Each triangle counts as $3/4$ vertices, 1 face, and $3/2$ edges, so adds $1/4$ to the total. Each square counts as $4/4$ vertices, 1 face, and $4/2$ edges, so adds 0 to the total. Such a shape must therefore have 8 triangles.

5. Find two trees with the same number of vertices that are homeomorphic but not isomorphic.

   The smallest such trees have 6 vertices each: one central vertex of degree 3 and three chains of lengths 1,1,3 and 1,2,2, respectively.

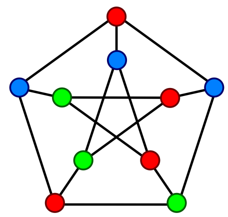
Planar Graphs

1. If a connected graph has n vertices and $n+2$ edges, then G is planar. For $n \geq 6$, this becomes false if we say $n+3$ instead of $n+2$.

   K_5 has 10 edges and 5 vertices while $K_{3,3}$ has 9 edges and 6 vertices. Any connected graph with n vertices containing a subgraph homeomorphic to either of these two must therefore have at least $n+3$ edges.

   $K_{3,3}$ is an example of a non-planar graph with n vertices and $n+3$ edges.

2. Show that the Petersen graph is not planar (find a subgraph homeomorphic to $K_{3,3}$).

   Solution on page 725 of Rosen.

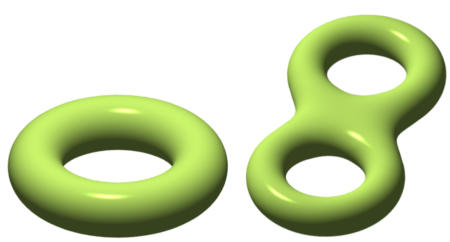
3. Find a bipartite graph G that is not planar but does NOT contain a subdivision of $K_{3,3}$. Your graph should have no more than 20 edges.

   Add a vertex to the middle of every edge of K_5. The resulting graph will be bipartite with 20 edges but does not contain $K_{3,3}$ since it is homeomorphic to K_5.

Bonus

1. Solve the problem that Matt Damon solved in *Godd Will Hunting*: "How many (non-isomorphic) homeomorphically irreducible trees are there on 10 vertices?" A tree is *homeomorphically irreducible* if it has no vertices of degree 2.

2. If $v - e + r = k$ for a tiling of some shape, then we can say that the *Euler characteristic* of the shape is k. For example, the Euler characteristic of a sphere is 2. Do some experimentation and find the Euler characteristics of the torus and double torus.

   The torus has Euler characteristic 0 (it can be tiled with squares, 4 to a vertex) and the double torus has Euler characteristic -2 (it can be tiled with 8 pentagons, 4 to a vertex).