Chapter 10.3-10.4: Paths and Isomorphisms

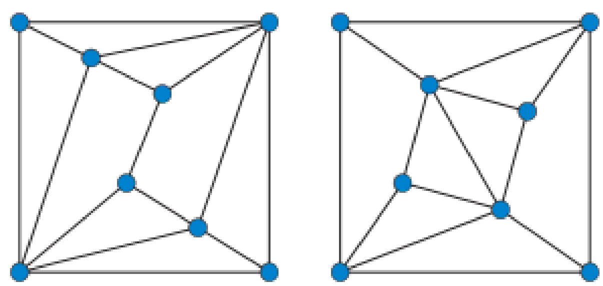
Monday, November 30

Definitions

- $\kappa(G)$: Vertex connectivity of G.
- $\lambda(G)$: Edge connectivity of G.

Isomorphisms

1. Show that C_5 and $\overline{C_5}$ are isomorphic.
   $\overline{C_5}$ is a five-pointed star, which is the same as a cycle of length 5 under the appropriate isomorphism.
2. Show that C_4 and $\overline{C_4}$ are not isomorphic.
   $\overline{C_4}$ has only 2 edges but C_4 has 4.
3. Find a graph G on 4 vertices such that G and $\overline{G}$ are isomorphic.
   The path of length 3 (P_3) is isomorphic to its complement.
4. Find all isomers (non-isomorphic graphs) of pentane (C_5H_{12}).
   There are three of them.
5. Show that the following two graphs are not isomorphic:



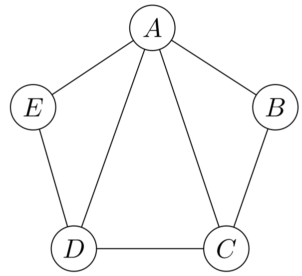
   The two vertices of degree 5 are adjacent in the first graph but not in the second.

Connectedness

1. Find a graph G such that $\kappa(G) < \lambda(G)$.
   Join two triangles at a single vertex.
2. Find a graph $G = (V, E)$ such that $\lambda(G) < \min_{v \in V} deg(v)$.
   Join two triangles by a single edge.

Adjacency Matrices

Here is a picture of a graph:



1. Draw the adjacency matrix A of the graph.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Find A^2. What do the diagonal elements of the matrix tell you?

$$A^2 = \begin{pmatrix} 4 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 2 \end{pmatrix}.$$

   The diagonal element A_{ii} tells you the degree of vertex i.

3. How can you use A^3 to count the triangles in a graph?

   Add up the diagonal elements of A^3, then divide by 3 since each triangle (cycle of length 3) is counted by 3 different vertices.

Proofs

1. Show that if G and $\overline{G}$ are isomorphic then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

   If they are isomorphic then G and $\overline{G}$ must have the same number of edges, thus the number of possible edges $\binom{n}{2}$ is even. So $2|n(n-1)/2$, and therefore $4|n(n-1)$. Since n and $n-1$ are relatively prime, this can only happen if $4|n$ or $4|(n-1)$.

2. Prove: if v has odd degree in G then there is some vertex w such that v and w are connected by a path.

   The Handshake Theorem (the number of vertices with odd degree is even) holds for the connected components of a graph as well as the entire graph. Thus if there is at least one vertex of odd degree in a component C then there must be another as well. Since C is connected, the two vertices are connected by a path.