Chapter 10.1-10.2: Graph Theory

Monday, November 13

Definitions

- K_n: the complete graph on n vertices
- C_n: the cycle on n vertices
- $K_{m,n}$ the complete bipartite graph on m and n vertices
- Q_n: the hypercube on 2^n vertices
- $H = (W, F)$ is a *spanning subgraph* of $G = (V, E)$ if H is a subgraph with the same set of vertices as G (i.e., $W = V$).
- $\overline{G} = (V, F)$ is the complement of $G = (V, E)$, where $(u, v) \in F$ if and only if $(u, v) \notin E$ for vertices $u, v \in V$.

Warmup

1. If $G = (V, E)$ is a simple graph, show that $|E| \leq \binom{n}{2}$.
2. How many triangles does the graph K_n contain?
3. Show that $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.
4. Show that number of graphs with n (labeled) vertices is $2^{\binom{n}{2}}$.
5. Draw K_4, C_5, $K_{2,4}$, and Q_3.

Graphs

1. How many edges does K_n have? What about C_n? $K_{m,n}$?
2. Let G have n vertices and m edges. How many induced subgraphs are there? How many spanning subgraphs are there?
3. How many spanning subgraphs of K_n are there with exactly m edges?
4. For which values of n is K_n bipartite? What about C_n?
5. Describe and count the edges of $\overline{K_n}, \overline{C_n}, \overline{K_{m,n}}$.

Pictures

1. Draw a directed graph on the 7 vertices $\{0, 1, \ldots, 6\}$ where (u, v) is an edge if and only if $v \equiv 3u \pmod{7}$.

2. Draw a directed graph on the 20 vertices $\{1, 2, \ldots, 20\}$ where (a, b) is an edge if and only if $a|b$. Organize the vertices in the clearest layout possible.

Proofs

1. Show that if G is bipartite then $|E| \leq \lfloor n^2/4 \rfloor$.

2. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree (Hint: pigeon-hole principle).

3. (Hard) If G is triangle-free, then $|E(G)| \leq \lfloor n^2/4 \rfloor$ (use inductions in increments of 2, delete both vertices connected by some edge for the inductive step).