Chapter 10.1-10.2: Graph Theory

Monday, November 13

Definitions

- K_n: the complete graph on n vertices
- C_n: the cycle on n vertices
- $K_{m,n}$ the complete bipartite graph on m and n vertices
- Q_n: the hypercube on 2^n vertices
- $H = (W, F)$ is a *spanning subgraph* of $G = (V, E)$ if H is a subgraph with the same set of vertices as G (i.e., $W = V$).
- $\overline{G} = (V, F)$ is the complement of $G = (V, E)$, where $(u, v) \in F$ if and only if $(u, v) \notin E$ for vertices $u, v \in V$.

Warmup

1. If $G = (V, E)$ is a simple graph, show that $|E| \leq \binom{n}{2}$.

   That's how many pairs of vertices there are.

2. How many triangles does the graph K_n contain?

   $\binom{n}{3}$, since each triangle is determined by 3 vertices.

3. Show that $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.

   Any of those possible pairs of vertices that aren't in $E(G)$ are in $E(\overline{G})$ and vice versa, so the sum is the total number of pairs of vertices.

4. Show that number of graphs with n (labeled) vertices is $2^{\binom{n}{2}}$.

   There are $\binom{n}{2}$ possible edges, and each one either is in the graph or isn't.

5. Draw K_4, C_5, $K_{2,4}$, and Q_3.

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Graphs

1. How many edges does K_n have? What about C_n? $K_{m,n}$?

   K_n has $\binom{n}{2}$ edges, C_n has n, $K_{m,n}$ has mn.

2. (★) Let G have n vertices and m edges. How many induced subgraphs are there? How many spanning subgraphs are there?

   There are 2^n induced subgraphs (all subsets of vertices) and 2^m spanning subgraphs (all subsets of edges).

3. How many spanning subgraphs of K_n are there with exactly m edges?

   $\binom{n}{m}$, since we fix all of the vertices and pick m edges.

4. For which values of n is K_n bipartite? What about C_n?

   K_n is bipartite only when $n \leq 2$. C_n is bipartite precisely when n is even.

5. Describe and count the edges of $\overline{K_n}, \overline{C_n}, \overline{K_{m,n}}$.

   Subtract the number of edges each of these graphs have from $\binom{n}{2}$ to get the number of edges in the complements.

Pictures

1. Draw a directed graph on the 7 vertices $\{0, 1, \ldots, 6\}$ where (u, v) is an edge if and only if $v \equiv 3u \pmod 7$.

   $0 \Rightarrow 0$, $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 6 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$.

2. Draw a directed graph on the 20 vertices $\{1, 2, \ldots, 20\}$ where (a, b) is an edge if and only if $a|b$. Organize the vertices in the clearest layout possible.

   The graph can be simplified if you assert that the relation is transitive and omit all of the edges that can be inferred from this property. Therefore you only have to draw the edges (a, b) such that $ap = b$ for some prime number p.

Proofs

1. Show that if G is bipartite then $|E| \leq \lfloor n^2/4 \rfloor$

   $K_{m,o}$ has mo edges in general, and this number is maximized when o and m are as close as possible. If n is even then $(n/2)(n/2) = n^2/4$, and if n is odd then we have $(n+1)/2 \cdot (n-1)/2 = (n^2-1)/4 < n^2/4$.

2. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree (Hint: pigeon-hole principle)

   Say the graph has n vertices with $n \geq 2$. The degree number of a vertex can be anywhere from 0 to n-1…n options in all. But if one vertex has degree 0 then no vertex can have degree $n-1$, because that would require it to share an edge with the degree 0 vertex. There are therefore at least (n-1) vertices with degrees between 1 and (n-2), so by the Pigeonhole principle two of them must have the same degree.

3. (Hard) If G is triangle-free, then $|E(G)| \leq \lfloor n^2/4 \rfloor$ (use inductions in increments of 2, delete both vertices connected by some edge for the inductive step).

   Base case: the inequality holds for $n = 1$ and $n = 2$.

   Inductive step: suppose the inequality holds for n, and consider a graph with $n+2$ vertices. Pick any edge (u, v). Because G is triangle-free, u and v can have no common neighbors (otherwise (u, v, w, u) would make a triangle), so each of the n remaining vertices is connected to at most one of u and v. Therefore if we delete u, v, and all edges connected to either of them, we will have deleted at most $n+1$ edges.

   The remaining graph has n vertices and by inductive hypothesis has at most $n^2/4$ edges, so when we add u and v back in we get that the graph G has at most $\frac{n^2}{4} + (n+1) = \frac{n^2+4n+4}{4} = \frac{(n+2)^2}{4}$ edges. The proof by induction is complete.