Chapter 9: Relations

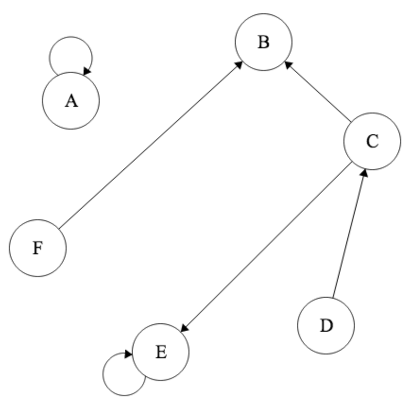
Wednesday, November 18

Warmup

Decide whether each of these relations is reflexive, symmetric, antisymmetric, or transitive:

1. $(a, b) \in R$ if $a > b$: Transitive, antisymmetric.
2. $(a, b) \in R$ if $a|b$: reflexive, transitive.
3. $(a, b) \in R$ if $5|a + b$: Symmetric.
4. $(a, b) \in R$ if $5|a - b$: Reflexive, symmetric, transitive.
5. $(a, b) \in R$ if $f(a) = f(b)$, for some fixed function f. Reflexive, symmetric, transitive.
6. $(a, b) \in R$ if a and b share a prime factor. Reflexive, symmetric.
7. $(a, b) \in R$ if $f(a) = g(b)$ for some fixed functions f and g. None.

Here is the digraph of a relation:



1. Draw the matrix that represents the relation.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Illustrate the smallest relation containing this one that is…

   (a) Reflexive: Include loops from every node to itself.

   (b) Transitive: Add a line from D to B.

   (c) Symmetric: Every arrow should be double-edged.

   (d) Reflexive, transitive, and symmetric: B,C,D,E,F are all connected directly to each other and themselves. A remains on its own.

Construction of the Rational Numbers

Suppose that we know what the integers are, as well as multiplication and addition. We would like the integers to have the property that for all $a \neq 0$, there is some $\overline{a}$ such that $\overline{a} \cdot a = 1$. Sadly, $\mathbb{Z}$ does not have this property, so we will make ourselves a set that does!
Define $S = (a, b) : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0$. Define a relation $\equiv$ on S as follows: $(a, b) \equiv (c, d)$ if $ad = bc$.

1. Prove that $\equiv$ is an equivalence relation. (This explains the notion of "equivalent fractions" without having to define division.)

   (a) Reflexive: $ab = ab$, so $(a, b) \equiv (a, b)$.

   (b) Symmetric: If $ad = bc$ then $bc = ad$, so if $(a, b) \equiv (c, d)$ then $(c, d) \equiv (a, b)$.

   (c) Transitive: Say $(a, b) \equiv (c, d)$ and $(c, d) \equiv (e, f)$. Then $ad = bc$ and $cf = de$, so $adf = bcf = bde$, meaning that $adf = bde$. Since $d \neq 0$, this implies that $af = be$, so $(a, b) \equiv (e, f)$.

2. (Remark): Since $\equiv$ is an equivalence relation, it partitions S into a set of equivalence classes. Define $\mathbb{Q}$ as the set of those classes.

3. Define "0" as the equivalence class $[(0, 1)]$. Find all pairs $(a, b) \in S$ that are equivalent to "0."

   $(0, b) \equiv (0, 1)$ for any $b \neq 0$.

4. Define "1," and find all elements in this class.

   $(a, a) \equiv (1, 1)$ for any $a \neq 0$.

5. Define multiplication as follows: $(a, b) \times (c, d) = (ac, bd)$. Now for the hard part: prove that muliplication is well-defined on the equivalence classes, so that if $(a_1, b_1) \equiv (a_2, b_2)$ and $(c_1, d_1) \equiv (c_2, d_2)$ then $(a_1, b_1) \times (c_1, d_1) \equiv (a_2, b_2) \times (c_2, d_2)$.

   We know that $a_1b_2 = b_1a_2$ and $c_1d_2 = d_1c_2$, and want to prove that $(a_1c_1, b_1d_1) \equiv (a_2c_2, b_2, d_2)$. This is true because we can multiply the left-hand sides and the right-hand sides of our two equations to get $a_1b_2c_1d_2 = b_1a_2d_1c_2$, which is exactly what we wanted.

6. Prove that if $a \in \mathbb{Q}$ is non-zero, then there exists $\overline{a}$ such that $\overline{a} \cdot a = 1$.

   If we have $(a, b) \in S$ with $(a, b) \neq 0$, then $a \neq 0$ (by a previous part), and so $(b, a) \in S$. Then $(a, b)(b, a) = (ab, ab) = 1$.

7. Define addition, and prove that it is well-defined on $\mathbb{Q}$.

   Much more tedious. Have fun!