Chapter 8.4-6: Generating Functions and Inclusion-Exclusion

Monday, November 16

Warmup

1. $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$
2. $\frac{1}{1-2x} = \sum_{i=0}^{\infty} (2x)^i$
3. $\frac{1}{1-x^2} = \sum_{i=0}^{\infty} x^{2i}$
4. $\sum_{i=0}^{\infty} (i+1)x^i = \left(\frac{1}{1-x}\right)^2$
5. If E and F are independent events, what is $p(E \cup F)$? $p(E \cup F) = p(E) + p(F) - p(E \cap F) = p(E) + p(F) - p(E)p(F)$.

Generating Functions

1. How many ways are there to make change for a dollar with pennies, nickels, dimes, quarters, and half-dollars? Do not find the answer, but explain how to get it using generating functions.

   The number of ways is equal to the coefficient of the x^{100} term in $\frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}\frac{1}{1-x^{50}}$.

2. Give eight cookies to three children so that each child gets between 1 and 4 cookies.

   Should be the x^8 coefficient of $(x + x^2 + x^3 + x^4)^3$, which is 12.

3. Find a function that generates the sequence $a_n = n \cdot 3^n$.

   $\left(\frac{1}{1-3x}\right)^2 - \frac{1}{1-3x}$.

More Inclusion-Exclusion

1. How many numbers between 1 and 60 are divisible by 2 or 3 or 5?

   By inclusion-exclusion, the number is $\frac{60}{2}+\frac{60}{3}+\frac{60}{5}-\frac{60}{6}-\frac{60}{10}-\frac{60}{15}+\frac{60}{30}=44$.

2. Four men check four hats, which at the end of the evening are returned to them randomly. For each number n between 0 and 4, find the probability that n of the men get their correct hat back.

   The chance of 0 men getting their hats back is the probability of a derangement, which is $(1-1+1/2-1/6+1/24)=3/8$.

   The chance of 1 man getting the right hat back is 4 times $(1/4)$ (decide which man gets the correct hat, then there is a 1/4 chance of that happening) times the probability of a 3-deragment, which is $(1-1+1/2-1/6)=1/3$.

   The chance of 2 men getting the right hat back is $\binom{4}{2}/12$ (there are $\binom{4}{2}$ ways to choose the 2 men that get their correct hats, and a $(1/4)\cdot(1/3)=1/12$ chance that they do get their correct hats) times the probability of a 2-deragment, so 1/4.

   There is no chance that only 1 man will get the wrong hat.

   The chance that all will get the right hat is 1/24.

   Note that $3/8+1/3+1/4+1/24=(9+8+6+1)/24=1$.

Relations

Decide whether each of these relations is reflexive, symmetric, antisymmetric, or transitive:

1. $(a,b)\in R$ if $a\geq b$. Reflexive, antisymmetric, transitive.
2. $(a,b)\in R$ if $a=b$. Reflexive, symmetric, transitive.
3. $(a,b)\in R$ if $ab=0$. Symmetric only.
4. $((a/b),(c,d))\in R$ if $ac=bd$. Reflexive, symmetric, transitive.

Draw a Venn Diagram with a circle for each of the three properties "reflexive," "symmetric," and "transitive." There are eight regions in this Venn Diagram. Find a relation that belongs in each region.
Good luck!