Chapters 1.4-1.6: Quantifiers and Rules of Inference — Solutions

Wednesday, September 2

Key Notes

- The order of the quantifiers matters!
- Nested negation: $\neg(\forall x)(\exists y)(P(x,y)) \equiv (\exists x)(\neg\exists y)(P(x,y)) \equiv (\exists x)(\forall y)(\neg P(x,y))$
- Conclusion C is valid given premises P as long as $P \rightarrow C$ is a tautology.

Warmup: Two-Player Game!

Let A = {“a”, “b”, “c”, . . . , “z”}, let W = $[(x, y) \in A^2|$ “xy” is an English word].

Game 1: Player 1 names a letter x and Player 2 names a letter y. If xy is a word then Player 1 wins. Who has a winning strategy?

Player 2 can win, pretty much just by making the second letter “q.”

Game 2: Now if xy is a word then Player 2 wins. Who has a winning strategy?

Player 1 can win by making the first letter “v.”

English to Quantifier

1. Every integer is either odd or even. $(\forall x \in \mathbb{Z})(x \in E \vee x \in O)$, where E = even integers, O = odd integers.
2. Either all integers or odd, or all integers are even. $(\forall x \in \mathbb{Z})(x \in O) \vee (\forall x \in \mathbb{Z})(x \in E)$. Note that two separate for-all statements are needed here.
3. The sum of any two even numbers is even. $(\forall x \in E)(\forall y \in E)(x + y \in E)$.
4. Everybody doesn’t like something, but nobody doesn’t like Sara Lee. (2 propositions)
   (a) $(\forall x)(\exists y)$(x does not like y)
   (b) $\neg(\exists x)$(x does not like Sara Lee), or
   (c) $(\forall x)$(x likes Sara Lee)
5. Everybody loves my baby, but my baby don’t love nobody but me. (2 propositions, maybe 3)
   (a) $(\forall x)$(L(x,B)), where B = my baby, L(x,y) = “x loves y”.
   (b) $(\forall x)(x \neq M \Rightarrow \neg L(B, x))$, where M = me
   (c) L(B,M)
6. The previous lyrics are from an old song popularized by Louis Armstrong. Prove: if we take “everybody” literally, then “my baby” and “me” must actually be the same person!
   (a) Everybody loves my baby (given)
   (b) Therefore, my baby loves my baby (instantiation)
   (c) My baby does not love anybody but me (given)
   (d) If my baby loves x then x is me (equivalent)
   (e) Since my baby loves “my baby”, we conclude that “my baby” = “me”.
7. Let M = {rock, paper, scissors}, and let $D(x, y)$ stand for “x defeats y.” How can you say “No move in rock-paper-scissors is guaranteed to win” in quantifier notation?
   $(\forall x \in M)(\exists y \in M)(\neg D(x, y))$.

Quantifier to English

Write each of the following statements in English and decide whether they are true or false. The domain is each case is the real numbers.

1. $(\forall x)(x^2 > 0)$

   For every real number x, x^2 is positive. FALSE: $x = 0$ is a counterexample.

2. $(\exists x)(\forall y)(x > y)$

   There is some x that is greater than all real numbers y. FALSE: x is never greater than itself, so we instead conclude $(\forall x)(\exists y)(x \leq y)$.

3. $(\forall x)(\exists y)(x > y)$

   For every x there is some y such that x is greater than y. TRUE: for any x, let $y = x - 1$.

4. $(\forall x > 0)(\exists y)(0 < y < x)$

   For every positive number x there is a smaller positive number y. TRUE: for any x, let $y = x/2$.

5. $(\forall x)(\exists y)(x + y = 10)$

   For any x there is a y such that $x + y = 10$. TRUE: let $y = 10 - x$.

6. $(\exists x)(\forall y)(x \cdot y = y)$

   There is some x such that $x \cdot y = y$ for every y. TRUE: $x = 1$ is the unique answer.

Write the negations of each of the above statements in quantifier notation and in English.
M: "You move"
S: "I will strike"
As a proposition, this is either $(M \Rightarrow S) \wedge (\neg M \Rightarrow S)$ or $(M \vee \neg M) \Rightarrow S$ (Show that the two are equivalent!)
For a rule of inference, we can use resolution, and the fact that $(S \vee S) \equiv S$:

$$\begin{array}{r l} M \Rightarrow S & \\ \underline{\neg M \Rightarrow S} & \\ \underline{S \vee S} & \text{(Resolution)} \\ S & \end{array}$$

Or going from the second variation we could say (since $M \vee \neg M$ is always true):

$$\begin{aligned} (M \vee \neg M) \Rightarrow S &\equiv \mathbf{T} \Rightarrow S \\ &\equiv S \end{aligned}$$

Deductive Reasoning

A traveler comes upon a snake, who says this: "If you move, I will strike. If you do not move, I will strike." Write this as a single compound proposition and show that it is logically equivalent to "I will strike." Which rule of inference applies here?
If Trevor gets stuck in traffic he will be late to work. If he is late to work he will be fired. He will not get stuck in traffic if *and only if* he takes the shortcut.
Which of the following conclusions are logically valid? Prove the ones that are.

1. If Trevor does not take the shortcut then he will be fired.

2. If Trevor takes the shortcut then he will not be fired.

3. If Trevor is not late to work then he took the shortcut.

Use the following notation:

1. S: takes the shortcut

2. T: stuck in traffic

3. L: late to work
4. F: gets fired

Use a visual to help deciding which conclusion is valid–based on the premises given, we can make the following:

$$\neg S \Leftrightarrow T \Rightarrow L \Rightarrow F$$

Taking the contrapositivies of the statements gives this:

$$S \Leftrightarrow \neg T \Leftarrow \neg L \Leftarrow \neg F$$

Based on these diagrams, conclusions 1 and 3 are valid but 2 is not. We can prove the first and third in two-column format:

$\neg S$	given
$\neg S \Rightarrow T$	given
T	Modus Ponens
$T \Rightarrow L$	given
L	Modus Ponens
$L \Rightarrow F$	given
F	Modus Ponens

We can prove the third similarly, relying on the contrapositives of the statements:

$\neg L$	given
$T \Rightarrow L$	given
$\neg T$	Modus Tollens
$\neg S \Rightarrow T$	given
S	Modus Tollens

If we lose the game, we will either cry or go out for ice cream (maybe both). Which conclusions are logically valid? Prove the ones that are.

1. If we do not cry and do not go out for ice cream, then we did not lose the game.
2. If we lose the game and do not go out for ice cream, then we will cry.
3. If we do not lose the game and we go out for ice cream, then we will not cry.

Use the following notation:

1. L: Lose the game
2. C: Cry
3. I: Go for ice cream

The original statement can be written as $L \Rightarrow (C \vee I)$. Its contrapositive is $(\neg C \wedge \neg I) \Rightarrow \neg L$, which is precisely conclusion 1.
Conclusion 3 is not valid: we said nothing about what would happen if we won the game. We might cry for joy.
To prove conclusion 2, we will take L and C as premises and then prove I:

Given	Goal
$L \Rightarrow (C \vee I)$ L $\neg I$	C

The proof then goes as follows:

L $L \Rightarrow (C \vee I)$	given
$C \vee I$ $\neg I$	Modus Ponens
C	Disjunctive Syllogism