

Chapters 1.1-1.3: Propositional Logic — Solutions

Monday, August 31

Key Topics

- Forward: $p \rightarrow q$; Inverse: $\neg p \rightarrow \neg q$; Converse: $q \rightarrow p$; Contrapositive: $\neg q \rightarrow \neg p$
- Forward \equiv contrapositive, inverse \equiv converse
- De Morgan's Law 1: $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- De Morgan's Law 2: $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- Definition of equivalence, tautology, contradiction

Warmup

All humans are mortal, for that is the way of life. All mortals like strawberries, for they are very tasty. Supposing that Odin is not mortal, which of the following *must* be true?

1. Odin is human.
2. Odin is not human. TRUE
3. Odin likes strawberries.
4. Odin does not like strawberries.
5. All humans like strawberries. TRUE
6. All strawberries like humans.

Show via truth tables that the propositions $p \rightarrow q$ and $\neg p \vee q$ are equivalent.

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

If-Then Statements

Rewrite the following phrases in the form “If P, then Q.” State the contrapositive, inverse, and converse. Decide whether the biconditional “P if and only if Q” is implied.

1. In order to run for president, you must be at least 35.
 - (a) If you are not at least 35 then you cannot run for president.
 - (b) If you are running for president then you are at least 35.
 - (c) The biconditional is not true.
2. Surrender or die!
 - (a) If you do not surrender then you will die.
 - (b) If you did not die then you surrendered.
 - (c) The biconditional is hopefully true.
3. No shirt, no shoes, no service.
 - (a) If you are not wearing a shirt or are not wearing shoes, then you will not get service.
 - (b) If you were given service then you were wearing a shirt and shoes.

- (c) You might be denied service for other reasons.
4. A watched pot never boils.
- (a) If you watch the pot then it will not boil.
 (b) If the pot boils then you were not watching it.
 (c) The pot also needs water and heat to boil.
5. You can't order alcohol unless you're over 21.
- (a) If you are not over 21 then you cannot order alcohol.
 (b) If you can order alcohol then you are over 21.
 (c) The converse is normally true, provided you have money.
6. The Warriors will win as long as they play well.
- (a) If the Warriors play well then they will win.
 (b) If they did not win then they did not play well.
 (c) It is possible for them to play badly and still win, but the events are at least correlated.
7. The Cubs lose whenever I watch them play.
- (a) If I watch the Cubs, they lose.
 (b) If the Cubs won, I was not watching them.
 (c) The converse does not hold, since the Cubs can still lose even when I am not watching.
8. There is a restaurant in Chicago that displays the following sign: "No shirt, no shoes, no pets, no bikes, no service." Please explain.
- If you do not have a shirt and shoes, or if you *do* have a pet or a bike, then you will not get service. The wording is rather confusing.

Equivalences

1. State the inverse, converse, and contrapositive of the following proposition: "If a number n is even and it is divisible by 3 then it is divisible by 6."
- (a) Inverse: If n is not even or not divisible by 3 then it is not divisible by 6.
 (b) Converse: If n is divisible by 6 then it is even and divisible by 3.
 (c) Contrapositive: If n is not divisible by 6 then it is not even or not divisible by 3.
 (d) In this case all four statements are true.
2. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent. Come up with an English example that illustrates the logic.

$$\begin{aligned}
 (p \rightarrow r) \wedge (q \rightarrow r) &\equiv (\neg p \vee r) \wedge (\neg q \vee r) \\
 &\equiv (\neg p \wedge \neg q) \vee r \\
 &\equiv \neg(p \vee q) \vee r \\
 &\equiv (p \vee q) \rightarrow r
 \end{aligned}$$

The equivalence from the second step to the third step is non-trivial, and it might be best to demonstrate it using truth tables (or Venn diagrams, but that won't come up until section 2.2).

English example:

- (a) If it rains I will stay inside and if it snows I will stay inside.
 - (b) If it rains or if it snows, I will stay inside.
3. Show that $((p \vee q) \wedge \neg p) \rightarrow q$ is a tautology. Come up with an English example that illustrates the logic.

$$\begin{aligned}
 ((p \vee q) \wedge \neg p) \rightarrow q &\equiv \neg((p \vee q) \wedge \neg p) \vee q \\
 &\equiv \neg(p \vee q) \vee \neg\neg p \vee q \\
 &\equiv \neg(p \vee q) \vee (p \vee q) \\
 &\equiv \mathbf{T}
 \end{aligned}$$

English example: If it is raining or snowing and it is not raining, then it is snowing.

Knights and Knaves

More Knights and Knaves! A recap: Knights always tell the truth; knaves always lie. If you meet two people, A and B, what can you deduce from their statements?

1. A says “We are both knights” and B says “A is lying!”

The statements are contradictory, so they cannot both be telling the truth. Thus A’s statement is false. A is a knave, and B (speaking truthfully) is therefore a knight.

2. A says “We are both knaves” and B says nothing.

A cannot be a knight since by his own testimony he would then be a knave. A must be a knave, and the only way for his statement to be false is for B to be a knight.

3. A says “I am a knave or B is a knight” and B says nothing.

A cannot be a knave since then his statement would be true. A must therefore be a knight, and for his statement to be true B must also be a knight.

Some of the islanders are also werewolves. Both knights and knaves can be werewolves. You are looking for a traveling partner and would rather go with a knight than a knave, but most of all you don’t want a werewolf for a partner (for obvious reasons). Say you meet three people and happen to know that exactly one is a werewolf. They tell you the following:

4. A: At least one of us is a knight.
 B: At least one of us is a knave.
 C: The werewolf is a knight.

Who should you pick as a partner based on their testimony?

If B were a knave then his statement would be true, which is a contradiction. B is therefore a knight. Then A is telling the truth and so is also a knight. But B (a knight) says that there is at least one knave, so that knave must be C. Since C is lying, the werewolf must be a knave. Therefore C is the werewolf.