Practice Midterm 2

Wednesday, October 28

Induction

1. Define a sequence a_n by $a_0 = 1$, $a_1 = 4$, and for $n \geq 2$, $a_n = 4a_{n-1} - 4a_{n-2}$. Prove: for all $n \geq 0$, $a_n = (n+1) \cdot 2^n$.
2. For a number $n \geq 0$, let $S(n)$ be the sum of the digits of n. Prove: repeatedly applying S to a positive number will eventually yield a number between 1 and 9.
3. Prove: for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$, $(1-x)^n \geq 1 - nx$.

Counting

1. 50 people go out to eat. Everyone orders either a hamburger or a salad (but not both). 15 people put mustard on their burgers, 25 put ketchup on their burgers, and 10 people put both ketchup and mustard on their burgers. How many people ordered a salad?
2. How many times must I roll a pair of dice in order to guarantee that I roll some number (the sum of the two dice) twice?
3. How many ways are there to put 3 red chairs and 4 blue chairs around a circular table if chairs of the same color are indistinguishable and two arrangements that differ only by rotating the table count as the same?
4. Evaluate the sum $\sum_{i=0}^{20} \binom{20}{i} (-1)^i 2^{20-i}$
5. How many ways are there to give 8 cookies to 4 friends if every friend must get at least 1 cookie?
6. How many ways are there to buy 7 fruit if my options are apples, bananas, and peaches?
7. How many ways are there to give 5 blue hats, 2 red hats, and 3 green hats to 10 friends?

Probability

1. I have 3 teal balls, 4 magenta balls, and 5 orange balls in a cauldron. If I draw 3 balls without replacement, what is the probability that I get 2 orange balls and 1 magenta? What if I draw 3 balls with replacement?
2. I have a coin that lands on heads 2/3 of the time and tails 1/3 of the time. If I flip the coin 4 times, what is the probability that I get 2 heads?
3. A fair coin and a loaded coin (p(heads) = 2/3) are sitting on a table. If I take a random coin and flip 2 heads out of 4, what is the chance that I took the fair coin?
4. Prove that if E and $\overline{E}$ are independent events then $p(E) = 0$ or $p(E) = 1$.
5. I roll two dice. If X is the sum of the rolls and Y is the product of the rolls, prove that X and Y are not independent random variables.
6. I roll a die, multiply the result by 3, then add 7. What is the expected value of my final number? What is the variance?
7. Prove that if E and F are independent random variables and $G = 2E+3$ then G and F are independent random variables.

8. I flip a coin. If it lands on tails, I get nothing. If it lands on heads, I roll a die and collect n dollars for rolling the number n. What is the expected value of the amount of money I will make?