Homework 8 Solutions

Math 55, DIS 101-102

6.3.12 [2 points]

How many bit strings of length 12 contain...

1. exactly three 1s?
   $\binom{12}{3} = 220$
2. at most three 1s?
   $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} = 1 + 12 + 66 + 220 = 299.$
3. at least three 1s?
   Take a shortcut by using the previous answer: $2^{12} - 66 - 12 - 1 = 4017$.
4. an equal number of 0s and 1s?
   $\binom{12}{6} = 924$

6.4.14 [2 points]

Show that each row of Pascal's Triangle is increasing until the middle and decreasing afterwards.

We can tell whether successive terms are increasing or decreasing by looking at the ratio $\binom{n}{k+1}/\binom{n}{k}$ and comparing this number to 1. With that:

$$\begin{aligned}\binom{n}{k+1}\Big/\binom{n}{k} &= \frac{n!}{(k+1)!(n-k-1)!}\frac{k!(n-k)!}{n!} \\ &= \frac{n!}{n!}\frac{k!}{(k+1)!}\frac{(n-k)!}{(n-k-1)!} \\ &= \frac{n-k}{k+1}.\end{aligned}$$

This quantity is equal to 1 if and only if $k = \frac{n-1}{2}$, which implies that n is odd and $k = \lfloor n/2 \rfloor$. The terms will then be increasing for smaller k and decreasing for larger k. If n is even then no two terms in a row will be equal, but $k = n/2$ will give the maximum value of $\binom{n}{k}$.

6.5.8 [2 points]

How many ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

This is a stars-and-bars problem with 12 selections and 20 dividers between donuts, so the number of ways is $\binom{12+20}{20} = \binom{12+20}{12} = 225792840$.

6.5.16 [2 points]

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$ with non-negative x_i such that...

1. $x_i > 1$ for all i?
   This condition is equivalent to $x_i \geq 2$, so substituting in new variables $y_i = x_i + 2$, $y_i \geq 0$ gives the new equation $\sum_i y_i = 17$. This is then a stars-and-bars problem whose answer is $\binom{17+5}{5} = 26334$.
2. $x_i \geq i$ for all i?
   Making the substitution $y_i = x_i + i$ (except for $y_5 = x_5 + 6$) gives the new equation $\sum_i y_i = 7$, so the number of solutions is $\binom{7+5}{5} = 792$.
   Note: many people misread this question because the problem had the inequality $x_5 > 5$ rather than $x_5 \geq 5$. Because this mistake was so simple to make, both $\binom{12}{5}$ and $\binom{13}{5}$ were counted as correct answers.

3. $x_1 \leq 5$?
   This is the same as all of the solutions ($\binom{29+5}{5}$) minus the ones where $x_1 \geq 6$ ($\binom{23+5}{5}$), so the answer is $\binom{34}{5} - \binom{28}{5} = 179976$.
4. $x_1 < 8$ and $x_2 > 8$?
   Substitute $y_2 = x_2 + 9$. Then the answer is all of the solutions ($\binom{20+5}{5}$) minus the ones where $x_1 \geq 8$ ($\binom{12+5}{5}$), which is 46942.