Homework 8 Solutions
Math 55, DIS 101-102

6.3.12 [2 points]

How many bit strings of length 12 contain...

1. exactly three 1s?
\[ \binom{12}{3} = 220 \]
2. at most three 1s?
\[ \binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} = 1 + 12 + 66 + 220 = 299. \]
3. at least three 1s?
Take a shortcut by using the previous answer: \(2^{12} - 66 - 12 - 1 = 4017.\)
4. an equal number of 0s and 1s?
\[ \binom{12}{6} = 924 \]

6.4.14 [2 points]

Show that each row of Pascal’s Triangle is increasing until the middle and decreasing afterwards.

We can tell whether successive terms are increasing or decreasing by looking at the ratio \(\binom{n}{k+1}/\binom{n}{k}\) and comparing this number to 1. With that:

\[
\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{k!(n-k)!}{n!} = \frac{n!}{(k+1)!} \cdot \frac{(n-k)!}{(n-k-1)!} = \frac{n-k}{k+1}.
\]

This quantity is equal to 1 if and only if \(k = \frac{n-1}{2},\) which implies that \(n\) is odd and \(k = \lfloor n/2 \rfloor.\) The terms will then be increasing for smaller \(k\) and decreasing for larger \(k.\) If \(n\) is even then no two terms in a row will be equal, but \(k = n/2\) will give the maximum value of \(\binom{n}{k}.\)

6.5.8 [2 points]

How many ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

This is a stars-and-bars problem with 12 selections and 20 dividers between donuts, so the number of ways is \(\binom{12+20}{20} = \binom{12+20}{12} = 225792840.\)

6.5.16 [2 points]

How many solutions are there to the equation \(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29\) with non-negative \(x_i\) such that...

1. \(x_i > 1\) for all \(i?\)
This condition is equivalent to \(x_i \geq 2,\) so substituting in new variables \(y_i = x_i + 2, y_i \geq 0\) gives the new equation \(\sum_i y_i = 17.\) This is then a stars-and-bars problem whose answer is \(\binom{17+5}{5} = 26334.\)
2. \(x_i \geq i\) for all \(i?\)
Making the substitution \(y_i = x_i + i\) (except for \(y_5 = x_5 + 6\)) gives the new equation \(\sum_i y_i = 7,\) so the number of solutions is \(\binom{7+5}{5} = 792.\)
Note: many people misread this question because the problem had the inequality \(x_5 > 5\) rather than \(x_5 \geq 5.\) Because this mistake was so simple to make, both \(\binom{12}{5}\) and \(\binom{13}{5}\) were counted as correct answers.
3. $x_1 \leq 5$?
   This is the same as all of the solutions \((\binom{29+5}{5})\) minus the ones where $x_1 \geq 6$ \((\binom{23+5}{5})\), so the answer is $\binom{34}{5} - \binom{28}{5} = 179976$.

4. $x_1 < 8$ and $x_2 > 8$?
   Substitute $y_2 = x_2 + 9$. Then the answer is all of the solutions \((\binom{20+5}{5})\) minus the ones where $x_1 \geq 8$ \((\binom{12+5}{5})\), which is 46942.