Homework 6 Solutions
Math 55, DIS 101-102

4.6.12 [2 points] Find all pairs \((a, b)\) such that the map \(p \mapsto ap + b \mod 26\) is its own inverse.

We want to find \((a, b)\) such that \(a(p + b) + b \equiv p \mod 26\) for all \(0 \leq p \leq 26\). If we set \(p = 0\) then this gives \(ab + b \equiv 0 \mod 26\). If we then set \(p = 1\), we get \(a^2 \equiv 1 \mod 26\).

The second congruence implies that \((a + 1)(a - 1) \equiv 0 \mod 26\), and the only solutions to this congruence are \(a \equiv \pm 1 \mod 26\). In the case \(a \equiv 1 \mod 26\) the congruence \((a + 1)b \equiv 0 \mod 26\) has the solutions \(b = 0, 13\), and in the case \(a \equiv -1 \mod 26\) any value of \(b\) will do.

Thus: the only solutions are \((1, 0)\) (which is the identity map), \((1, 13)\) (which shifts every number by 13), and \((-1, b)\) for any \(b\).

5.1.4 [2 points]

1. The statement \(P(1)\) is that \(1^3 = \left(\frac{1 \cdot 2}{2}\right)^2\).
2. \(\left(\frac{1 \cdot 2}{2}\right)^2 = (1)^2 = 1 = 1^3\), so \(P(1)\) is true.
3. The inductive hypothesis is that \(\sum_{i=1}^{n} i^3 = \left(\frac{i(i+1)}{2}\right)^2\).
4. For the inductive step, we need to prove that \(\sum_{i=1}^{n+1} i^3 = \left(\frac{(i+1)(i+2)}{2}\right)^2\).
5.
\[
\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n + 1)^3
\]
\[
= \text{IH} \frac{n^2 \cdot (n + 1)^2}{4} + \frac{4(n+1)(n+1)^2}{4}
\]
\[
= \frac{(n + 1)^2(n^2 + 4n + 4)}{4}
\]
\[
= \frac{(n + 1)^2(n + 2)^2}{4}
\]
6. We know that \(P(1)\) is true and that \(P(n) \rightarrow P(n+1)\) for any \(n \in \mathbb{N}\), so the principle of induction implies that \(P(n)\) is true for all natural numbers \(n \geq 1\).

5.1.10 [2 points]

1. Define \(S(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}\), and observe \(S(1) = \frac{1}{2}\), \(S(2) = \frac{2}{3}\), \(S(3) = \frac{3}{4}\). Conjecture that \(S(n) = \frac{n}{n+1}\) for all \(n \in \mathbb{N}\).
2. Base case: \(S(1) = \frac{1}{2}\). Check!
Inductive step: Suppose that $S(n) = \frac{n}{n+1}$. Then

\[ S(n + 1) = \sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} = IH \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{n+1}{n+2}. \]

This completes the proof by induction.

5.2.14 [2 points]

Define $S(n)$ as the sum of all products resulting from splitting the piles of stones. If $n = 1$ then there are no products, so the sum of all products is 0. If $n = 2$ then we can only split the pile into two piles of one stone each, so $S(2) = 1 \cdot 1 = 1$. Also observe the recursive definition of the function $S(a + b) = S(a) + S(b) + a \cdot b$. Begin the proof by induction:

Base case: If $n = 1$ then $S(n) = 0 = \frac{0(0-1)}{2}$.

Inductive step: Suppose that $S(k) = \frac{k(k-1)}{2}$ for all $1 \leq k < n$. Then if we split the pile into two piles $n = a + b$, we must necessarily have $1 \leq a, b < n$, and so can apply the inductive hypothesis. Therefore:

\[ S(n) = S(a + b) = S(a) + S(b) + a \cdot b = IH \frac{a(a-1)}{2} + \frac{b(b-1)}{2} + 2ab = a(a-1) + ab + b(b-1) + ab = \frac{a(a + b - 1) + b(a + b - 1)}{2} = \frac{(a + b)(a + b - 1)}{2} = \frac{n(n-1)}{2}. \]

This completes the proof by induction.