Homework 6 Solutions

Math 55, DIS 101-102

4.6.12 [2 points] Find all pairs (a, b) such that the map $p \mapsto ap + b \bmod 26$ is its own inverse.

We want to find (a, b) such that $a(ap + b) + b \equiv p \pmod{26}$ for all $0 \leq p \leq 26$. If we set $p = 0$ then this gives $ab + b = (a + 1)b \equiv 0 \pmod{26}$. If we then set $p = 1$, we get $a^2 \equiv 1 \pmod{26}$.

The second congruence implies that $(a + 1)(a - 1) \equiv 0 \pmod{26}$, and the only solutions to this congruence are $a \equiv \pm 1 \pmod{26}$. In the case $a \equiv 1 \pmod{26}$ the congruence $(a + 1)b \equiv 0 \pmod{26}$ has the solutions $b = 0, 13$, and in the case $a \equiv -1 \pmod{26}$ any value of b will do.

Thus: the only solutions are $(1, 0)$ (which is the identity map), $(1, 13)$ (which shifts every number by 13), and $(-1, b)$ for any b.

5.1.4 [2 points]

1. The statement $P(1)$ is that $1^3 = \left(\frac{1 \cdot 2}{2}\right)^2$.
2. $\left(\frac{1 \cdot 2}{2}\right)^2 = (1)^2 = 1 = 1^3$, so $P(1)$ is true.
3. The inductive hypothesis is that $\sum_{i=1}^{n} i^3 = \left(\frac{i \cdot (i+1)}{2}\right)^2$.
4. For the inductive step, we need to prove that $\sum_{i=1}^{n+1} i^3 = \left(\frac{(i+1) \cdot (i+2)}{2}\right)^2$.
5.
$$\begin{aligned} \sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\ &=^{IH} \frac{n^2 \cdot (n+1)^2}{4} + \frac{4(n+1)(n+1)^2}{4} \\ &= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \end{aligned}$$
6. We know that $P(1)$ is true and that $P(n) \to P(n+1)$ for any $n \in \mathbb{N}$, so the principle of induction implies that $P(n)$ is true for all natural numbers $n \geq 1$.

5.1.10 [2 points]

1. Define $S(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}$, and observe $S(1) = \frac{1}{2}$, $S(2) = \frac{2}{3}$, $S(3) = \frac{3}{4}$. Conjecture that $S(n) = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.
2. Base case: $S(1) = \frac{1}{2}$. Check!

Inductive step: Suppose that $S(n) = \frac{n}{n+1}$. Then

$$\begin{aligned}
S(n+1) &= \sum_{i=1}^{n+1} \frac{1}{i(i+1)} \\
&= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\
&=^{IH} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}.
\end{aligned}$$

This completes the proof by induction.

5.2.14 [2 points]

Define $S(n)$ as the sum of all products resulting from splitting the piles of stones. If $n = 1$ then there are no products, so the sum of all products is 0. If $n = 2$ then we can only split the pile into two piles of one stone each, so $S(2) = 1 \cdot 1 = 1$. Also observe the recursive definition of the function $S(a + b) = S(a) + S(b) + a \cdot b$. Begin the proof by induction:

Base case: If $n = 1$ then $S(n) = 0 = \frac{0 \cdot (0-1)}{2}$.

Inductive step: Suppose that $S(k) = \frac{k(k-1)}{2}$ for all $1 \leq k < n$. Then if we split the pile into two piles $n = a+b$, we must necessarily have $1 \leq a, b < n$, and so can apply the inductive hypothesis. Therefore:

$$\begin{aligned}
S(n) &= S(a+b) \\
&= S(a) + S(b) + a \cdot b \\
&=^{IH} \frac{a(a-1)}{2} + \frac{b(b-1)}{2} + \frac{2ab}{2} \\
&= \frac{a(a-1) + ab + b(b-1) + ab}{2} \\
&= \frac{a(a+b-1) + b(a+b-1)}{2} \\
&= \frac{(a+b)(a+b-1)}{2} \\
&= \frac{n(n-1)}{2}.
\end{aligned}$$

This completes the proof by induction.