Homework 3 Solutions

Math 55, DIS 101-102

(2.2.20) [2 points] Suppose A and B are sets with $A \subseteq B$. Prove:

1. $A \cup B = B$.

   Proof: If $x \in B$ then $x \in A$ or $x \in B$, so $x \in A \cup B$. Therefore $B \subset A \cup B$.

   If $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in B$ then we are done. If $x \in A$ then $x \in B$ because $A \subseteq B$. Either way, $x \in B$. Therefore, $A \cup B \subset B$.

   Since $B \subset A \cup B$ and $A \cup B \subset B$, we conclude that if $A \subset B$ then $A \cup B = B$.

2. $A \cap B = A$.

   Slick proof: We know from the previous problem that $A \cup B = B$. So take complements and use De Morgan's Law to get $\overline{A} \cap \overline{B} = \overline{B}$. But $A \subset B$ is equivalent to $\overline{B} \subset \overline{A}$, so this proves the statement "If $\overline{B} \subset \overline{A}$ then $\overline{A} \cap \overline{B} = \overline{B}$". Renaming the sets appropriately gives the desired result.

   Standard proof: If $x \in A \cap B$ then $x \in A$ and $x \in B$, so $x \in A$.

   If $x \in A$ then $x \in B$ because $A \subseteq B$. So if $x \in A$ then $x \in A$ and $x \in B$, meaning $x \in A \cap B$. Therefore, $A \subset A \cap B$.

   Since $A \cap B \subset A$ and $A \subset A \cap B$, we conclude that if $A \subset B$ then $A \cap B = A$.

The most common mistake here was to show only one direction (say, $B \subseteq A \cup B$) and not the other ($A \cup B \subseteq B$).

(2.3.12) [2 points] Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.

1. $f(n) = n - 1$ is one-to-one. Proof: If $f(a) = f(b)$ then $a - 1 = b - 1$, so $a - b$.
2. $f(n) = n^2 + 1$ is not one-to-one. Proof: $f(-1) = 2 = f(1)$.
3. $f(n) = n^3$ is one-to-one. A variety of proofs will work for this one:
   (a) Since the derivative of the real-valued function $g(x) = x^3$ is always non-negative and is only zero at a single point, $g(x)$ (and therefore $f(x)$) is an increasing function. Increasing functions are one-to-one.
   (b) For any n, $(n+1)^3 - n^3 = 3n^2 + 3n + 1 = 3(n + 1/2)^2 + 1/4 \geq 1/4 > 0$, so $f(n) = n^3$ is an increasing function. Increasing functions are one-to-one. Alternately, $3n^2 + 3n + 1$ is always positive because the leading coefficient is positive and the equation $3n^2 + 3n + 1 = 0$ has no real roots.
   (c) Suppose $a^3 = b^3$. Then $0 = a^3 - b^3 = (a-b)(a^2+ab+b^2)$, so either $a = b$ or $a^2+ab+b^2 = 0$. If the second case is true, then (supposing without loss of generality that $|b| \geq |a|$) $a^2+ab+b^2 \geq a^2 + b^2 - |ab| \geq a^2 + b^2 - b^2 = a^2 \geq 0$, so the two are equal only if $a = 0$ and $b = 0$. Therefore, if $a^3 = b^3$ then $a = b$, so f is one-to-one.
4. $f(n) = \lceil n/2 \rceil$ is not one-to-one, since $f(1) = 1 = f(2)$.

(2.3.44) [2 points]

1.

$$\begin{aligned} f^{-1}(S \cup T) &= [x \in A : f(x) \in S \cup T] \\ &= [x \in A : f(x) \in S \vee f(x) \in T] \\ &= [x \in A : f(x) \in S] \cup [x \in A : f(x) \in T] \\ &= f^{-1}(S) \cup f^{-1}(T) \end{aligned}$$

   Alternate proof, though the differences are only superficial:

For any $x \in A$,

$$\begin{aligned} x \in f^{-1}(S \cup T) &\Leftrightarrow f(x) \in S \cup T \\ &\Leftrightarrow f(x) \in S \vee f(x) \in T \\ &\Leftrightarrow x \in f^{-1}S \vee x \in f^{-1}T \\ &\Leftrightarrow x \in f^{-1}(S) \cup f^{-1}(T). \end{aligned}$$

Similarly to the proof involving set equalities, many students here only showed inclusion in one direction (e.g. $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$) and not the other, but asserted that this was enough to show that the sets were equal. In the case of the proof immediately above, linking the propositions with if-and-only-if statements ($\Leftrightarrow$) is enough to show that the sets are equal, but having if-then statements ($\Rightarrow$) is not.

2.

$$\begin{aligned} f^{-1}(S \cup T) &= [x \in A : f(x) \in S \cap T] \\ &= [x \in A : f(x) \in S \wedge f(x) \in T] \\ &= [x \in A : f(x) \in S] \cap [x \in A : f(x) \in T] \\ &= f^{-1}(S) \cap f^{-1}(T) \end{aligned}$$

(2.4.14) [0 points]

1. $a_n = 3$: $a_0 = 3, a_{n+1} = a_n$.
2. $a_n = 2n$: $a_0 = 0$, $a_{n+1} = a_n + 2$.
3. $a_n = 2n + 3$: $a_0 = 3$, $a_n = 2 + a_{n-1}$.
4. $a_n = 5^n$: $a_0 = 1$, $a_n = 5 \cdot a_{n-1}$.
5. $a_n = n^2$: $a_0 = 0, a_{n+1} = a_n + 2n + 1$.
6. $a_n = n^2 + n$: $a_0 = 0, a_n = 2n + a_{n-1}$.
7. $a_n = n + (-1)^n : a_0 = 1, a_{n+1} = a_n + 1 - 2 \cdot (-1)^n$.
8. $a_n = n! : a_0 = 1, a_n = n \cdot a_{n-1}$.

(2.5.10) [2 points]

1. Finite difference: $A = [0, 1], B = [0, 1), A - B = \{1\}$.
2. Countable difference: $A = \mathbb{R}, B = \mathbb{R} - \mathbb{Q}, A - B = \mathbb{Q}$.
3. Uncountable difference: $A = [0, 1], B = [1, 2], A - B = [0, 1)$.

(2.5.30) [0 points]

The set of all triplets of integers (a, b, c) is equal to $\mathbb{Z}^3$, which is countable. Each triplet defines a quadratic equation with at most 2 real solutions, so the set of real numbers that are solutions of such equations is at most $|2\mathbb{Z}^3| = |\mathbb{Z}|$, and is therefore countable.