Homework 1 Solutions

Math 55, DIS 101-102

(1.5.20) [2 points]

1. It was most common for people to write "The product of two negative integers is positive" as $(\forall x)(\forall y)((x < 0) \wedge (y < 0) \rightarrow xy > 0)$. This is correct, but it can be stated more elegantly by restricting the scopes of the quantifiers for x and y: $(\forall x < 0)(\forall y < 0)(xy > 0)$. Since x and y have the same restriction, we can even combine the forall statements and get $(\forall x, y < 0)(xy > 0)$.
2. The most common answer for "The average of two positive integers is positive" was $(\forall x)(\forall y)((x > 0) \wedge (y > 0) \rightarrow (\frac{x+y}{2} > 0))$. Similar to the previous problem, a simpler answer would by $(\forall x, y > 0)(\frac{x+y}{2} > 0)$.
3. This problem gave people the most trouble. First, the correct way to interpret "The difference of two negative integers is not necessarily negative" is that there is a *counterexample* to the claim that the difference of any two negative integers is negative... in other words, $\neg(\forall x)(\forall y)((x < 0) \wedge (y < 0) \rightarrow x - y < 0)$. This answer is technically correct but unsatisfactory because it is hard to interpret.

   It is better to carry the negation through to the end, using the fact that $\neg(p \rightarrow q) \equiv p \wedge \neg q$ (a counterexample to "If p then q" is a case where p is true but q is not):

$$\begin{aligned}
\neg(\forall x)(\forall y)((x < 0) \wedge (y < 0) \rightarrow x - y < 0) &\equiv (\exists x)(\exists y)((x < 0) \wedge (y < 0) \wedge \neg(x - y < 0)) \\
&\equiv (\exists x)(\exists y)((x < 0) \wedge (y < 0) \wedge x - y \geq 0)) \\
&\equiv (\exists x, y < 0)(x - y \geq 0)
\end{aligned}$$

4. Some people left this one as $(\forall x)(\forall y)(\neg(|x + y| > |x| + |y|))$. This is again correct but ungainly, and a better answer would be $(\forall x, y)(|x + y| \leq |x| + |y|)$.

(1.5.39) [2 points]

1. Claim: $\forall x, y(x^2 = y^2 \rightarrow x = y)$. Any case where $x = -y$ with $x, y \neq 0$ is a counterexample.
2. Claim: $\forall x \exists y(y^2 = x)$. Since the domain is the integers, any value of x that is not a perfect square (and especially $x < 0$)) is a counterexample.
3. Claim: $\forall x, y(xy \geq x)$. This is equivalent to the claim that $x(y - 1) \geq 0$, which is false whenever $x < 0$ or $y < 1$ (but not both).

(1.7.8) [2 points]

1. Best solution: use contradiction. Suppose that n and $n + 2$ are both perfect squares, so write $n = a^2$ and $n + 2 = b^2$ with $b > a \geq 0$. Then $2 = n + 2 - n = b^2 - a^2 = (b + a)(b - a)$. Since $b > a \geq 0$, $b + a$ and $b - a$ are both positive. The only possible factorization is therefore $b + a = 2, b - a = 1$, which has solutions $b = \frac{3}{2}, a = \frac{1}{2}$. This contradicts the assertion that both a and b were integers, and therefore if n is a perfect square then $n + 2$ is not.
2. Also okay: If $n = 0$ then $n + 2 = 2$ is not a perfect square. So suppose $n = a^2$ with $a \geq 1$. Then the next smallest square is $(a+1)^2 = a^2 + 2a + 1 \geq n + 2 + 1 = n + 3$, so $n + 2$ cannot be a perfect square.

The biggest problem that people faced with this one was making assertions without providing sufficient (or any) evidence. In particular, saying that the difference between 2 squares can never be 2 because $0^2 = 0, 1^2 = 1, 2^2 = 4 \ldots$ is not a satisfactory proof, because giving a handful of examples can never prove a claim that is supposed to apply to all integers.

(1.7.22) [0 points]

Some people wrote out all 8 combinations of black/blue socks. This works but does not extend well to larger problems. A better solution is to use contraposition (or contradiction): If no pair of matching socks has been drawn, then there is at most 1 blue and 1 black sock, therefore at most 2 socks. By the contrapositive, if 3 (or more) socks have been drawn then there is a matching pair.

(1.8.7) [0 points]

Most people were okay with this one, but I wanted to share a slick proof of the triangle inequality that a former student found: just square both sides!

$$\begin{aligned} |x+y| \le |x|+|y| &\Leftrightarrow (x+y)^2 \le (|x|+|y|)^2 \\ &\Leftrightarrow x^2+2xy+y^2 \le x^2+2|x||y|+y^2 \\ &\Leftrightarrow 2xy \le 2|x||y| \\ &\Leftrightarrow xy \le |xy|, \end{aligned}$$

and the final statement is true. The first and second inequalities are equivalent because both sides of the equation are non-negative. This proof does rely on two smaller identities: $x^2 = |x|^2$, and $|xy| = |x||y|$.

(1.8.23) [2 points]

This is related to what we discussed in class on Wednesday: if you start at the conclusion and give a series of equations that end up at a true statement, this only implies that your conclusion is true if the equations are connected by "only if" or "if and only if" statements. A good (backward) proof should make this relation explicit:

$$\begin{gathered} \frac{2xy}{x+y} \le \sqrt{xy} \\ \Leftrightarrow 2\sqrt{xy} \le x+y \\ \Leftrightarrow 4xy \le x^2+2xy+y^2 \\ \Leftrightarrow 0 \le x^2-2xy+y^2 \\ \Leftrightarrow 0 \le (x-y)^2. \end{gathered}$$

Without the explicit assertion that successive statements are equivalent, the default reading of the above proof would be "If $\frac{2xy}{x+y} \le \sqrt{xy}$ then $0 \le (x-y)^2$," which is true but not what we want to prove (we want to prove the converse).

Remember that proofs are meant to be read. If you cannot read your proof aloud so that your logic is understandable, it needs work.

Also for fun, one student found a nice proof of the inequality that also involves squaring both sides: let $A = \frac{2xy}{x+y}$ and let $B = \sqrt{xy}$. Since $A, B \ge 0$, $A \le B \Leftrightarrow A^2 \le B^2 \Leftrightarrow B^2 - A^2 \ge 0$. The final inequality is true because

$$
\begin{aligned}
B^2 - A^2 &= \sqrt{xy}^2 - \left(\frac{2xy}{x+y}\right)^2 \\
&= xy - \frac{4x^2y^2}{(x+y)^2} \\
&= \frac{xy(x^2+2xy+y^2)}{(x+y)^2} - \frac{xy(4xy)}{(x+y)^2} \\
&= \frac{xy(x^2-2xy+y^2)}{(x+y)^2} \\
&= \frac{xy}{(x+y)^2}(x-y)^2 \\
&\geq 0
\end{aligned}
$$

Moral of the story: There is rarely just one correct way to prove a statement.