Homework 11 Solutions

Math 55, DIS 101-102

8.1.12 [2 points]

1. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take 1, 2, or 3 steps at a time.

$$S(n) = S(n-1) + S(n-2) + S(n-3)$$

2. What are the initial conditions?

$$S(-2) = S(-1) = 0; S(0) = 1.$$

3. How many ways can this person climb a flight of eight stairs?

n	$S(n)$
0	1
1	1
2	2
3	4
4	7
5	13
6	24
7	44
8	81

8.1.20 [2 points]

A bus driver pays all tolls using only nickels and dimes.

1. Find a recurrence relation for the number of ways to pay a toll of n cents where the order in which the coins are used matters.

$$T(n) = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ T(n-5) + T(n-10) & n > 0 \end{cases}.$$

2. How many ways can the driver pay a toll of 45 cents?

n	$T(n)$
0	1
5	1
10	2
15	3
20	5
25	8
30	13
35	21
40	34
45	55

Note that this is just the Fibonacci sequence.

8.2.8 [2 points]

Assume that the number of lobsters caught per year is the average of the numbers caught in the previous two years.

1. Find a recurrence relation for $\{L_n\}$.
   $L_n = L_{n-1}/2 + L_{n-2}/2$.
2. Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.
   The associated polynomial is $x^2 - x/2 - 1/2 = 0$ with roots $x = 1, -1/2$, so the solution is of the form $a + b(-1/2)^n$. Taking the initial conditions into account gives

$$\begin{aligned} a - b/2 &= 100,000 \\ a + b/4 &= 300,000 \\ b &= 800,000/3 \\ a &= 700,000/3 \end{aligned}$$

   The solution is therefore $L_n = 700,000/3 + 800,000/3(-1/2)^n$. This means that as $n \to \infty$, $L_n \to 700,000/3 \approx 233,000$.

8.2.24 [2 points] Consider the recurrence relation $a_n = 2a_{n-1} + 2^n$.

1. Show that $a_n = n2^n$ is a solution of the recurrence relation.

$$2a_{n-1} + 2^n = 2(n-1)2^{n-1} + 2^n = n2^n = a_n.$$

2. Use Theorem 5 to find all solutions of this recurrence relation.
   The general solution to the homogeneous equation is $a_n = c\dot{2}^n$, so all solutions are of the form $c \cdot 2^n + n2^n$.
3. Find the solution with $a_0 = 2$.
   Solving gives $c = 2$, so the solution is $a_n = 2 \cdot 2^n + n2^n = (n+2) \cdot 2^n$.