Homework 10 Solutions

Math 55, DIS 101-102

7.4.6 [2 points] What is the expected value when a $1 lottery ticket is bought in which the purchaser wins $10 million if the ticket contians the six winning numbers chosen from the set $\{1 \ldots 50\}$ and nothing otherwise?

The expected value is $\frac{10,000,000}{\binom{50}{6}} - 1 \approx -.37$. You can expect to lose 37 cents whenever you buy a ticket.

7.4.10 [2 points] Suppose we flip a fair coin until it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?

For $2 \leq n \leq 5$, the number of ways to arrange the two tails so that the second one is on the final flip is $(n-1)$, so the probability of stopping after n flips for $n < 6$ is $(n-1)/2^n$. The sum of these probabilities is 13/16, so there is a 3/16 chance of stopping at $n = 6$. The expected number of times we flip the coin is then

$$2 \cdot \frac{1}{4} + 3 \cdot \frac{2}{8} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{4}{32} + 6 \cdot \frac{3}{16} = 15/4 = 3.75$$

7.4.12 [2 points] Suppose we roll a fair die until a 6 comes up.

1. What is the probability that we roll the die n times?

   We would have to roll $n-1$ numbers that are not 6 followed by a 6, so the probability is $(5/6)^{n-1} \cdot (1/6)$.

2. What is the expected number of times we roll the die?

$$E(N) = \sum_{n=1}^{\infty} n \cdot (5/6)^{n-1}(1/6) = (1/6)\left(\frac{1}{1-(5/6)}\right)^2 = 6.$$

   This is a nice answer since after 6 rolls we would expect to have rolled exactly one 6.

7.4.19 [2 points]

Let X be the number on the first die when two dice are rolled and Y be the sum of the two numbers. Show that $E(X)E(Y) \neq E(XY)$.

$$\begin{aligned}
E(XY) &= \frac{1}{36}\sum_{n=1}^{6}\sum_{m=1}^{6} n(n+m) \\
&= \frac{1}{36}\sum_{n=1}^{6} 6n^2 + 21n \\
&= \frac{987}{36} \\
&= \frac{329}{12} \\
&> 24.5 \\
&= \frac{7}{2} \cdot 7 \\
&= E(X)E(Y)
\end{aligned}$$

The equation $E(X)E(Y) \neq E(XY)$ does not necessarily hold because these two random variables are not independent.

Another solution: avoid doing arithmetic whenever possible and instead use linearity of expectations:

$$\begin{aligned} E(XY) &= E(X(X+W)) \\ &= E(X^2) + E(XW) \\ &> E(X)^2 + E(XW) \\ &= E(X)^2 + E(X)E(W) \\ &= E(X)E(X+W) \\ &= E(X)E(Y), \end{aligned}$$

using the fact that $E(X^2) > E(X)^2$ because X has non-zero variance.

Another solution: define the covariance $Cov(X,Y) = E(XY) - E(X)E(Y)$ (see problems 7.4.44-46 in Rosen) and show that this number is not zero.

$$\begin{aligned} Cov(X, X+W) &= Cov(X,X) + Cov(X,W) \\ &= Cov(X,X) \\ &= Var(X) \\ &> 0. \end{aligned}$$

since X (the roll of a single die) is not constant.