Review Problems for Final: Answers

1. Graph $f(x) = \frac{x^2+4x+3}{x-2}$.

   Graphs are attached after the rest of the solutions.

   (a) Vertical asymptote at $x = 2$.
   (b) Since $x^2 + 4x + 3 = (x + 3)(x + 1)$, zeroes are at $x = -1, -3$.
   (c) $f(x) = x + 6 + \frac{15}{x-2}$, so the slant asymptote is $x + 6$.
   (d) $f'(x) = 1 - 15/(x - 2)^2 = 0$ when $x = 2 \pm \sqrt{15} \approx -2, 2$.
   (e) $f''(x) = 30/(x - 2)^3$, which is positive when $x > 2$ and negative when $x < 2$. Thus the critical point at $x = 2 - \sqrt{15}$ is a local max and the point at $x = 2 + \sqrt{15}$ is a local min. The function is always concave down when $x < 2$ and concave up when $x > 2$.

2. Graph $f(x) = \frac{1}{x} + \sqrt{x}$.

   (a) Domain is $x > 0$, with a vertical asymptote at $x = 0$.
   (b) $f(x)$ is always positive, $\lim_{x\to\infty} f(x) = \infty$.
   (c) $f'(x) = -1/x^2 + \frac{1}{2\sqrt{x}} = 0$ when $x = \sqrt[3]{4}$. Since this is the only critical point and $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = \infty$, we know that it must be a local minimum. Alternatively, show this by the first derivative test or the (following) second derivative test.
   (d) $f''(x) = 2/x^3 - \frac{1}{4x^{3/2}} = 0$ when $x = 4$. Plugging in a point or two will show that the graph is concave up for $x < 4$ and concave down for $x > 4$.
   (e) Additional note: as $x \to \infty$, $1/x \to 0$, so the graph will look more and more like $\sqrt{x}$.

3. Graph $f(x) = \sqrt{x^2 - 2}$.

   (a) Domain: $x^2 - 2 \geq 0$, so $x \geq \sqrt{2}$ or $x \leq -\sqrt{2}$.
   (b) $f(x)$ is always positive, and $\lim_{x\to\pm\infty} f(x) = \infty$.
   (c) $f(x)$ is even, thus symmetrical about the $y-$axis.
   (d) Check for a slant asymptote: $\lim_{x\to\infty} f(x)/x = \lim \sqrt{x^2 - 2}/x = \lim \sqrt{1 - 2/x^2} = 1$, so there is a slant asymptote with slope 1.
   (e) $\lim_{x\to\infty} f(x) - mx = \lim \sqrt{x^2 - 2} - x = \lim(\sqrt{x^2 - 2} - x)(\sqrt{x^2 - 2} + x)/(\sqrt{x^2 - 2} + x) = \lim -2/(\sqrt{x^2 - 2} + x) = 0$, so the slant asymptote as $x \to \infty$ is $y = x$.
   (f) Since the function is even, the slant asymptote for $x \to -\infty$ is $y = -x$.
   (g) $f'(x) = \frac{x}{\sqrt{x^2-2}}$, which is positive for $x > 0$ and negative for $x < 0$. Also, $\lim_{x\to\pm\sqrt{2}} f(x) = \pm\infty$, so the graph has vertical tangent lines at $x = \pm\sqrt{2}$.
   (h) $f''(x) = \frac{\sqrt{x^2-2}-x^2/\sqrt{x^2-2}}{x^2-2} = \frac{x^2-2-x^2}{(x^2-2)^{3/2}} = \frac{-2}{(x^2-2)^{3/2}}$, which is always negative. Thus the graph is always concave down.

4. *Find a constant k such that* $\lim_{x\to 0} \frac{\sin(x)+kx}{x^3}$ *exists. Find the limit.*

   Both numerator and denominator have limit zero as $x \to 0$, so using L'Hospita's Rule we get

$$\lim_{x\to 0} \frac{\sin(x) + kx}{x^3} = \lim_{x\to 0} \frac{\cos(x) + k}{3x^2}$$

   The denominator again has a limit of zero, so the limit of the whole function can exists ONLY if the numerator also has a limit of zero (so that the whole limit is indeterminate). Thus $k = -1$. We can now use L'Hospital's rule two more times, which yields

$$\lim_{x\to 0} \frac{\cos(x) + k}{3x^2} = \lim_{x\to 0} \frac{-\sin(x)}{6x} = -1/6$$

5. *Use Newton's Method to fin the next two terms for*

   - $x^2 - x - 1 = 0$, $x_0 = 1$
     Our iteration function is $N(x) = x - f(x)/f'(x) = x - \frac{x^2-x-1}{2x-1}$. This gives $x_1 = N(x_0) = 1 - (-1/1) = 2$ and $x_2 = N(x_1) = 2 - (1/3) = 5/3$.
   - $ax - b = 0$, $x_0 = c$
     Our iteration function is $N(x) = x - f(x)/f'(x) = x - (ax - b)/a = b/a$, which is independent of the starting point x. Therefore $x_1 = x_2 = b/a$, and so Newton's method converges to the correct solution in just one step.

6. *A 10m wire is cut into two pieces, one bent to make a square and the other an equilateral triangle. Maximize/minimize the combined area of the two shapes.*

   Let S be the length of wire used to make the square, so that $(10-S)$ is the length used for the triangle. The square has side lengths $S/4$, so has area $S^2/16$. The triangle has side lengths $(10 - S)/3$ and area $\frac{(10-S)^2\sqrt{3}}{36} = \frac{\sqrt{3}}{36}(100 - 20S + S^2)$.

   The derivative of the area with respect to S is therefore $A' = S(2+2\sqrt{3}/36) - 20\sqrt{3}/36$, which is zero when $S = \frac{10\sqrt{3}}{36+\sqrt{3}} \approx .459$. The second derivative is $A'' = 2 + \sqrt{3}/18$, which is always positive, so this point is the unique minimum.

   Checking the endpoints shows that the total area is maximized when $S = 10$, so that all of the wire is used toward making the square.

7. Maximize/minimize $g(x) = \int_{\sin(x)}^{\cos(x)} t\, dt$

   By the First Fundamental Theorem of Calculus, $g'(x) = \cos(x)(-\sin(x)) - \sin(x)\cos(x) = -2\sin(x)\cos(x) = -\sin(2x)$, which is zero when $x = k\pi/2$. $g''(x) = -2\cos(2x)$, which is positive ($g(x)$ is at a local minimum) when $x = \pi/2 + k\pi$ and negative ($g(x)$ is at a local maximum) when $x = k\pi$.

   Alternatively, solve the integral and get $g(x) = \frac{1/2}{t}^2\Big|_{\sin(x)}^{\cos(x)} = \frac{1}{2}(\cos(x)^2 - \sin(x)^2) = \frac{1}{2}\cos(2x)$, and find $g'(x)$ from there.

8. Write the (Right hand) Riemann sums for the areas of triangles with the following vertices, and find the corresponding integrals:

   - $(0, 0), (1, 0), (1, 1)$
     $A = \int_0^1 x\, dx$, so $a = 0, b = 1$.
     $\Delta x = (b - a)/n = 1/n$
     $x_i = a + i\Delta x = i/n$
     Using right hand sums, the height of the i-th rectangle is $f(x_i) = i/n$, so the Riemann sum is $\lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n (i/n)(1/n) = \lim_{n\to\infty} \sum_{i=1}^n i/n^2$.
   - $(1, 0), (2, 0), (2, 1)$
     $A = \int_1^2 (x - 1)\, dx$, so $a = 1, b = 2, f(x) = x - 1$.
     $\Delta x = (b - a)/n = 1/n$
     $x_i = a + i\Delta x = 1 + i/n$
     Riemann sum is $\lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n ((1+i/n)-1)(1/n) = \lim_{n\to\infty} \sum_{i=1}^n i/n^2$, same as the first sum.
   - $(0, 0), (1/2, 0), (1/2, 2)$
     $A = \int_0^{1/2} 4x\, dx$, so $a = 0, b = 1/2, f(x) = 4x$.
     $\Delta x = (b - a)/n = 1/2n$.
     $x_i = a + i\Delta x = i/2n$.

Riemann sum is $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} 4(i/2n)(1/2n) = \lim_{n\to\infty}\sum_{i=1}^{n} i/n^2$, same as the first two sums.

The second integral relates to the first via the substitution $u = x - 1$, and the third relates to the first by the substitution $u = 2x$.

9. Find the volume of the region bounded by $y = 1 - x^2$ and $y = 0$, rotated around...

   - $x = 1$

     The shape ranges in y-value from 0 to 1 and in x-value from -1 to 1.

     Use cylindrical shells, integrating with respect to x. The height h is equal to $|y_1 - y_2| = (1 - x^2) - 0 = 1 - x^2$, and the radius is equal to the distance of x from the axis of rotation, so $r = |1 - x| = 1 - x$.

     We then get $V = \int_{x=-1}^{1} 2\pi rh\,dx = 2\pi \int_{x=-1}^{1}(1-x)(1-x^2)\,dx = 8\pi/3$.

   - $y = 1$

     Again integrate with respect to x, this time using washers. The outer radius is a constant $R = 1 - y_2 = 1$ and the inner radius is $r = |1 - y_1| = 1 - (1 - x^2) = x^2$.

     The volume is then $V = \int_{x=-1}^{1} \pi(R^2 - r^2)\,dx \int_{x=-1}^{1} \pi(1^2 - (x^2)^2)\,dx = \pi \int_{x=-1}^{1} 1 - x^4 = 8\pi/5$.

10. *Prove that e^x and e^{-x} intersect exactly once.*

    To prove that they intersect at least once:

    (a) Observe that $e^0 = e^{-0} = 1$.

    (b) e^x, e^{-x} are both continuous. $e^{-1} = 1/e < e = e^{-(-1)}$, and $e^1 = e > 1/e = e^{-1}$, so by the Intermediate Value Theorem they intersect somewhere in the interval $[-1, 1]$.

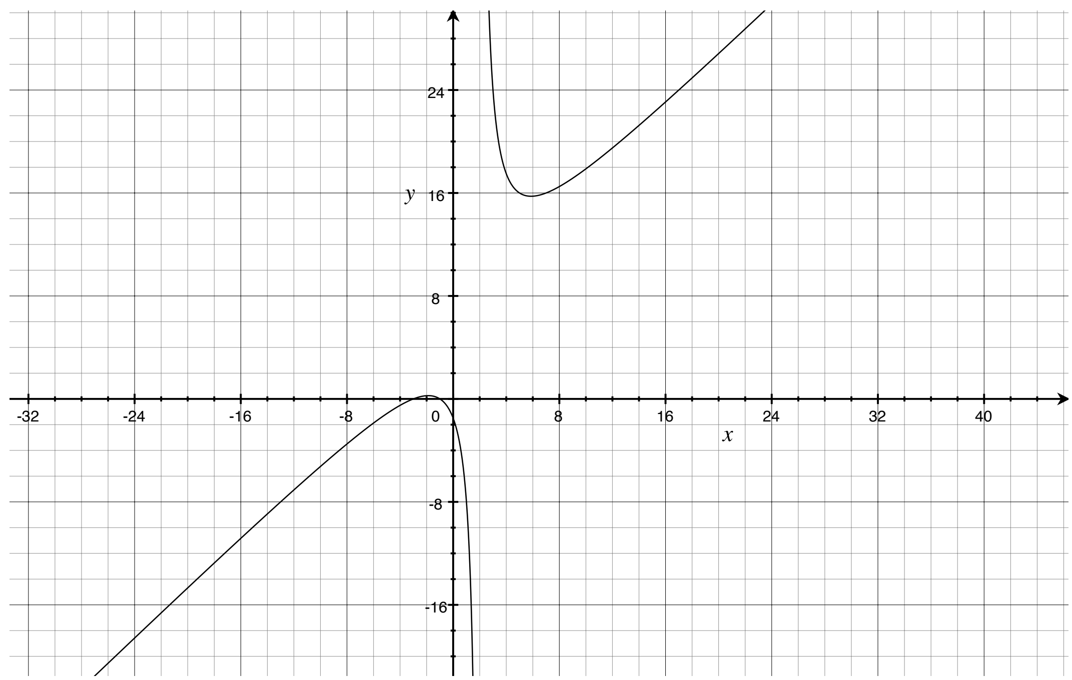
    (c) Let $f(x) = e^x - e^{-x}$. $f(-1) < 0 < f(1)$, so by the IVT there is an $x \in [-1, 1]$ such that $f(x) = 0$.

    (d) Or observe that f is an odd function, thus $f(0) = 0$.

    To show that they do not intersect again: $\frac{d}{dx}e^x = e^x > 0$, so (as a corollary of the Mean Value Theorem) e^x is increasing. $\frac{d}{dx}e^{-x} = -e^{-x} < 0$, so (again by the MVT) e^{-x} is decreasing.

11. *Find the volume of a pyramid with height 1 and a square base of side length 1.*

    This problem appears as an example in chapter 6 (6.2?) of the textbook; see the text for the solution.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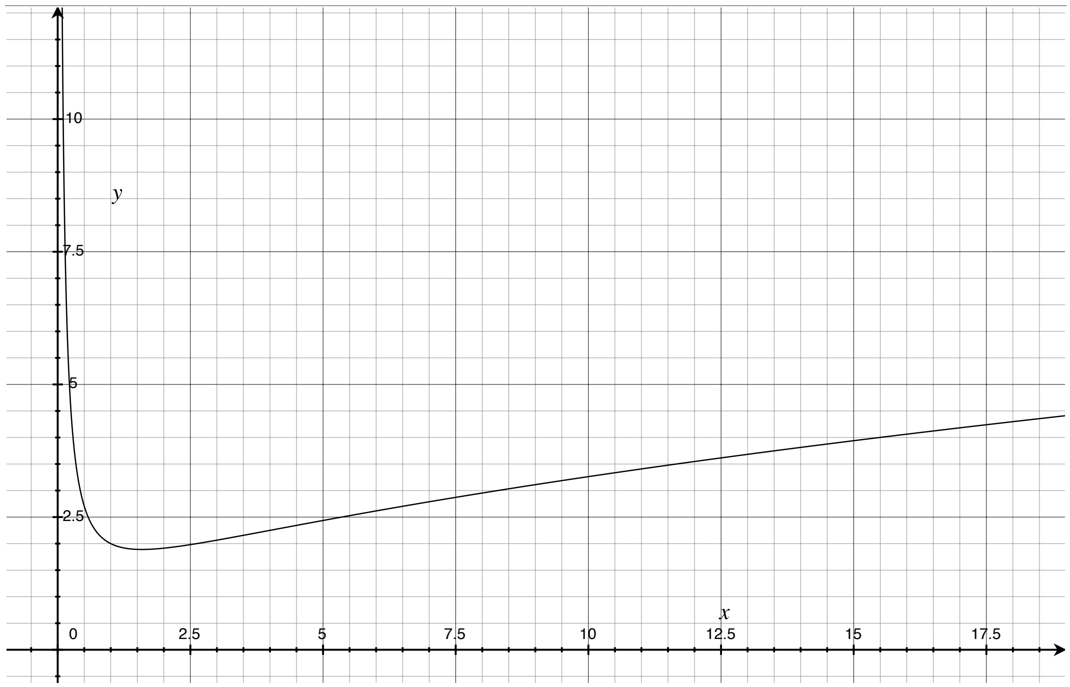

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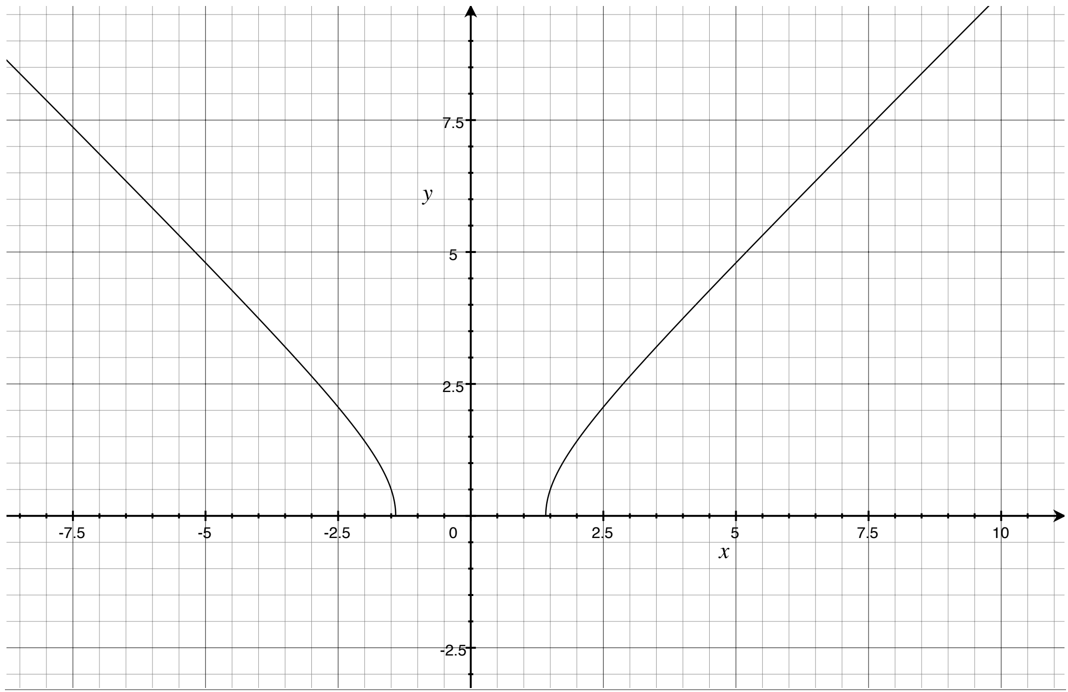

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