Mock Exam for Final

1. Graph $y = \frac{-2x^2+x+1}{x+3}$.

   (a) Vertical asymptote at $x = -3$.

   (b) $-2x^2 + x + 1 = -(x-1)(2x+1)$, so zeroes at $x = 1, -1/2$.

   (c) Long division gives $\frac{-2x^2+x+1}{x+3} = -2x + 7 - \frac{20}{x+3}$, so the slant asymptote is $y = -2x + 7$.

   (d) $f'(x) = -2 + 20/(x+3)^2$, which is zero when $x = 3 \pm \sqrt{10} \approx -6, 0.1$.

   (e) $f''(x) = -40/(x+3)^3$, which is negative when $x > -3$ and positive when $x < -3$. Thus the graph is concave up when $x < -3$ and the point near $x = -6$ is a local minimum. The graph is concave down when $x > -3$ and the point near $x = 0$ is a local maximum.

2. Rotate the area enclosed by $y = x^2$ and $y = \sqrt{x}$ around the line $x = -2$, and find the volume of the resulting solid.

   Note that the enclosed area is bounded by two curves that intersect at $(0,0)$ and $(1,1)$.

   Strategy 1: Shell method. Integrate along the x-axis. Formulae for radius and height given by $r = |-2-x| = x+2$, $h = |x^2 - \sqrt{x}| = \sqrt{x} - x^2$. Therefore

$$\begin{aligned} V &= 2\pi \int_{x=0}^{1} rh\,dx \\ &= 2\pi \int_{x=0}^{1} (x+2)(\sqrt{x} - x^2)\,dx \\ &= 2\pi \int_{x=0}^{1} 2\sqrt{x} + x\sqrt{x} - 2x^2 - x^3\,dx \\ &= 2\pi(4/3x^{3/2} + 2/5x^{5/2} - 2x^3/3 - x^4/4)|_0^1 \\ &= 49\pi/30 \end{aligned}$$

   Strategy 2: Washer method. Integrate along the y-axis. New functions given by $x = \sqrt{y}, x = y^2$. Outer radius is therefore $R = 2 + \sqrt{x}$, inner radius is $r = 2 + x^2$. Then get the integral

$$\begin{aligned} V &= \pi \int_{y=0}^{1} (R^2 - r^2)\,dy \\ &= \pi \int_{y=0}^{1} (2+\sqrt{x})^2 - (2+x^2)^2\,dy \\ &= 49\pi/30 \end{aligned}$$

3. Consider a table with a rectangular center and a semicircular cap at each end. The rectangle has length L and width 10, and the two semicircular ends both have radius 5 (thus fit squarely on the side of length 10). What value of L will maximize the perimeter-to-area ratio (P/A) of the table?

   The perimeter of the table is given by $P = 2L + 10\pi$, and the area is given by $A = 10L + 25\pi$. Therefore the ratio of the two is $P/A = \frac{2L+10\pi}{10L+25\pi} = \frac{1}{5}\frac{2L+10\pi}{2L+5\pi} = \frac{1}{5}(1 + \frac{5\pi}{2L+5\pi})$.

   The derivative is then $(P/A)' = \frac{-2\pi}{(2L+5\pi)^2}$, which is always negative. The only possible critical point is therefore at $L = 0$, and since the function is decreasing (the derivative is always negative), $L = 0$ is a global maximum.

4. Find $\lim_{x\to 0} \frac{\int_{\cos(x)}^{1} t\ln(t)\,dt}{x}$.

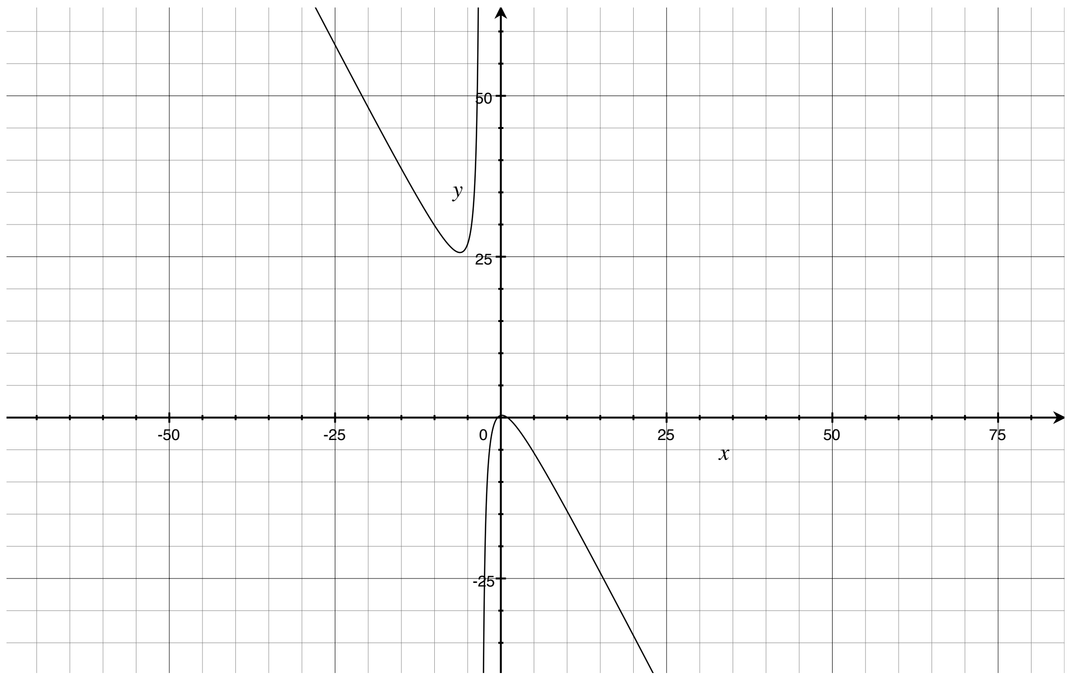
   First note that $\lim_{x\to 0} \int_{\cos(x)}^{1} t\ln(t)\,dt = \int_{1}^{1} t\ln(t)\,dt = 0$, so both numerator and denominator converge to zero and the limit is indeterminate. Use L'Hospital's rule to get

$$\begin{aligned}\lim_{x\to 0} \frac{\int_{\cos(x)}^{1} t\ln(t)\,dt}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}\int_{\cos(x)}^{1} t\ln(t)\,dt}{1} \\ &= \lim_{x\to 0} -\cos(x)\ln(\cos(x))(-\sin(x)) \\ &= (-1)*(0)*(0) \\ &= 0\end{aligned}$$

5. Find $\lim_{n\to\infty} \sum_{i=1}^{n} \sin(i\pi/n)\pi/n$.

   Try to interpret as a (Right-hand) Riemann sum of the form $\sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$. The most reasonable guess is $\Delta x = \pi/n$, after which the simplest values to choose for a and b are $a = 0; b = \pi$. (Any value for a is technically possible as long as $b = a + \pi$, but this choice is the simplest).

   Then $f(x_i) = f(i\pi/n) = \sin(i\pi/n)$, so the most reasonable guess for f is $f(x) = \sin(x)$. We now have $f(x) = \sin(x)$, $a = 0, b = \pi$, so by the limit definition of the integral our sum is equal to $\int_{x=0}^{\pi} \sin(x)\,dx$. Using the Fundamental Theorem of Calculus to take the integral, we find that $\int_{x=0}^{\pi} \sin(x)\,dx = -\cos(\pi) - (-\cos(0)) = 2$.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