42. $f(x) = \begin{cases} x+1 & \text{if } x \leqslant 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } x \geqslant 3 \end{cases}$

43. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2-x & \text{if } x > 1 \end{cases}$

44. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\[2ex] \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

45. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

end HW7

46. Find the values of a and b that make f continuous everywhere.

HW8

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

47. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

48. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

49. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

50. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

51–54 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

51. $x^4 + x - 3 = 0, \quad (1, 2)$

52. $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

53. $e^x = 3 - 2x, \quad (0, 1)$

54. $\sin x = x^2 - x, \quad (1, 2)$

55–56 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

55. $\cos x = x^3$

56. $\ln x = 3 - 2x$

57–58 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

57. $100e^{-x/100} = 0.01x^2$

58. $\arctan x = 1 - x$

59. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

60. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 59 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use $\boxed{6}$ to show that this is true.

61. Prove that cosine is a continuous function.

62. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

63. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

64. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

65. Is there a number that is exactly 1 more than its cube?

66. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

9 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 74.)

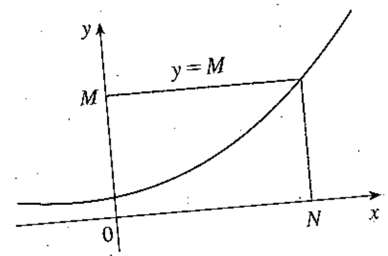


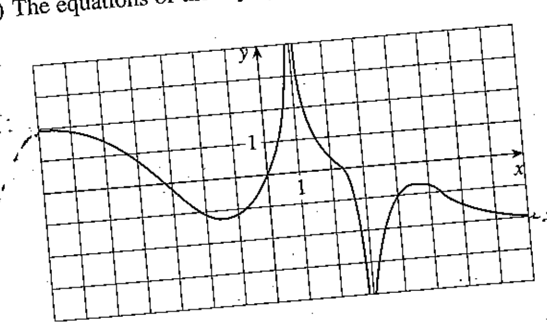
FIGURE 19
$\lim_{x \to \infty} f(x) = \infty$

2.6 Exercises

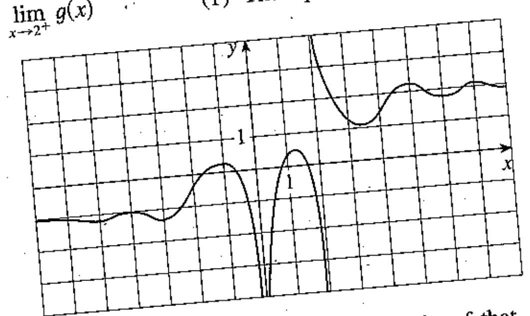
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$
(c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 3} f(x)$
(e) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
(c) $\lim_{x \to 0} g(x)$ (d) $\lim_{x \to 2^-} g(x)$
(e) $\lim_{x \to 2^+} g(x)$ (f) The equations of the asymptotes



5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $\lim_{x \to 0} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = 5$, $\lim_{x \to \infty} f(x) = -5$

6. $\lim_{x \to 2} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = \infty$, $\lim_{x \to -2^-} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = 0$, $f(0) = 0$

7. $\lim_{x \to 2} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$

8. $\lim_{x \to \infty} f(x) = 3$, $\lim_{x \to 2^-} f(x) = \infty$, $\lim_{x \to 2^+} f(x) = -\infty$, f is odd

9. $f(0) = 3$, $\lim_{x \to 0^-} f(x) = 4$, $\lim_{x \to 0^+} f(x) = 2$, $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to 4^-} f(x) = -\infty$, $\lim_{x \to 4^+} f(x) = \infty$, $\lim_{x \to \infty} f(x) = 3$

10. $\lim_{x \to 3} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 2$, $f(0) = 0$, f is even

11. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x \to \infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$

14. $\lim_{x \to \infty} \sqrt{\dfrac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

15–38 Find the limit or show that it does not exist.

15. $\lim_{x \to \infty} \dfrac{3x - 2}{2x + 1}$

16. $\lim_{x \to \infty} \dfrac{1 - x^2}{x^3 - x + 1}$

17. $\lim_{x \to -\infty} \dfrac{x - 2}{x^2 + 1}$

18. $\lim_{x \to -\infty} \dfrac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

19. $\lim_{t \to \infty} \dfrac{\sqrt{t} + t^2}{2t - t^2}$

20. $\lim_{t \to \infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

21. $\lim_{x \to \infty} \dfrac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

22. $\lim_{x \to \infty} \dfrac{x^2}{\sqrt{x^4 + 1}}$

23. $\lim_{x \to \infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

24. $\lim_{x \to -\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

25. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$

26. $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2x}\right)$

27. $\lim_{x \to \infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

28. $\lim_{x \to \infty} \sqrt{x^2 + 1}$

29. $\lim_{x \to \infty} \dfrac{x^4 - 3x^2 + x}{x^3 - x + 2}$

30. $\lim_{x \to \infty} (e^{-x} + 2\cos 3x)$

31. $\lim_{x \to -\infty} (x^4 + x^5)$

32. $\lim_{x \to -\infty} \dfrac{1 + x^6}{x^4 + 1}$

33. $\lim_{x \to \infty} \arctan(e^x)$

34. $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

35. $\lim_{x \to \infty} \dfrac{1 - e^x}{1 + 2e^x}$

36. $\lim_{x \to \infty} \dfrac{\sin^2 x}{x^2 + 1}$

37. $\lim_{x \to \infty} (e^{-2x}\cos x)$

38. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

39. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.
(b) Use a table of values of $f(x)$ to guess the value of the limit.
(c) Prove that your guess is correct.

40. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
(c) Find the exact value of the limit.

41–46 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

41. $y = \dfrac{2x + 1}{x - 2}$

42. $y = \dfrac{x^2 + 1}{2x^2 - 3x - 2}$

43. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

44. $y = \dfrac{1 + x^4}{x^2 - x^4}$

45. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$

46. $y = \dfrac{2e^x}{e^x - 5}$

47. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

48. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).
(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

49. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x\to\pm\infty} f(x) = 0, \quad \lim_{x\to 0} f(x) = -\infty, \quad f(2) = 0,$$
$$\lim_{x\to 3^-} f(x) = \infty, \quad \lim_{x\to 3^+} f(x) = -\infty$$

50. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

51. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x\to -1} f(x) = 2$. Evaluate
(a) $f(0)$ (b) $\lim_{x\to\infty} f(x)$

52–56 Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

52. $y = 2x^3 - x^4$
53. $y = x^4 - x^6$
54. $y = x^3(x + 2)^2(x - 1)$
55. $y = (3 - x)(1 + x)^2(1 - x)^4$
56. $y = x^2(x^2 - 1)^2(x + 2)$

57. (a) Use the Squeeze Theorem to evaluate $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$.
(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

58. By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.
(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.
(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

59. Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

60. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:
(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even
Then use these sketches to find the following limits.
(a) $\lim_{x\to 0^+} x^n$ (b) $\lim_{x\to 0^-} x^n$
(c) $\lim_{x\to\infty} x^n$ (d) $\lim_{x\to -\infty} x^n$

61. Find $\lim_{x\to\infty} f(x)$ if, for all $x > 1$,

$$\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x - 1}}$$

62. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

63. In Chapter 9 we will be able to show, under certain assumptions, that the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and v^* is the *terminal velocity* of the raindrop.
(a) Find $\lim_{t\to\infty} v(t)$.
(b) Graph $v(t)$ if $v^* = 1$ m/s and $g = 9.8$ m/s^2. How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

64. (a) By graphing $y = e^{-x/10}$ and $y = 0.1$ on a common screen, discover how large you need to make x so that $e^{-x/10} < 0.1$.
(b) Can you solve part (a) without using a graphing device?

65. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left|\frac{3x^2 + 1}{2x^2 + x + 1} - 1.5\right| < 0.05$$

66. For the limit

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 7 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

67. For the limit

$$\lim_{x\to -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = -2$$

illustrate Definition 8 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

68. For the limit

$$\lim_{x\to\infty} \frac{2x + 1}{\sqrt{x + 1}} = \infty$$

illustrate Definition 9 by finding a value of N that corresponds to $M = 100$.

69. (a) How large do we have to take x so that $1/x^2 < 0.0001$?
(b) Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

70. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
(b) Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

71. Use Definition 8 to prove that $\displaystyle\lim_{x\to -\infty} \frac{1}{x} = 0$.

72. Prove, using Definition 9, that $\lim_{x\to\infty} x^3 = \infty$.

73. Use Definition 9 to prove that $\lim_{x\to\infty} e^x = \infty$.

74. Formulate a precise definition of

$$\lim_{x\to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x\to -\infty} (1 + x^3) = -\infty$$

75. Prove that

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f(1/t)$$

and

$$\lim_{x\to -\infty} f(x) = \lim_{t\to 0^-} f(1/t)$$

if these limits exist.

2.7 Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in Section 2.1. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

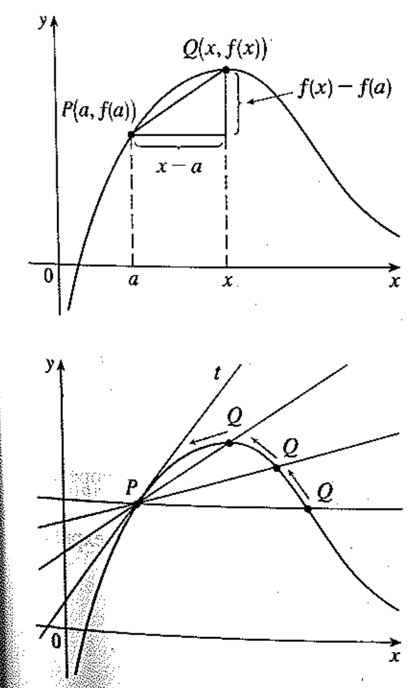


FIGURE 1

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 1 in Section 2.1.