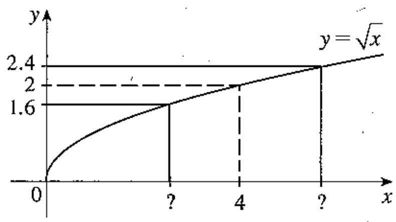
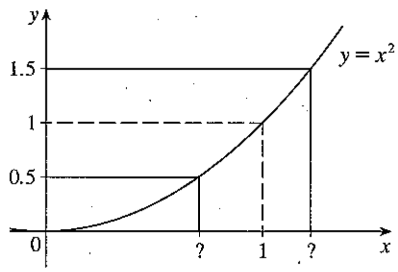
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \quad |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \tfrac{1}{2}$$



5. Use a graph to find a number δ such that

$$\text{if} \quad \left| x - \frac{\pi}{4} \right| < \delta \quad \text{then} \quad |\tan x - 1| < 0.2$$

6. Use a graph to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad \left| \frac{2x}{x^2 + 4} - 0.4 \right| < 0.1$$

7. For the limit

$$\lim_{x \to 2} (x^3 - 3x + 4) = 6$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. Given that $\lim_{x \to \pi/2} \tan^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 1000$ and (b) $M = 10{,}000$.

10. Use a graph to find a number δ such that

$$\text{if} \quad 5 < x < 5 + \delta \quad \text{then} \quad \frac{x^2}{\sqrt{x - 5}} > 100$$

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.

(a) What radius produces such a disk?

(b) If the machinist is allowed an error tolerance of $\pm 5 \text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?

(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

(a) How much power is needed to maintain the temperature at 200°C?

(b) If the temperature is allowed to vary from 200°C by up to ± 1°C, what range of wattage is allowed for the input power?

(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.

(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x \to 2}(5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

15. $\displaystyle\lim_{x \to 3} \left(1 + \tfrac{1}{3}x\right) = 2$

16. $\displaystyle\lim_{x \to 4} (2x - 5) = 3$

17. $\displaystyle\lim_{x \to -3} (1 - 4x) = 13$

18. $\displaystyle\lim_{x \to -2} (3x + 5) = -1$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\displaystyle\lim_{x \to 1} \frac{2 + 4x}{3} = 2$

20. $\displaystyle\lim_{x \to 10} \left(3 - \tfrac{4}{5}x\right) = -5$

21. $\displaystyle\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = 5$

22. $\displaystyle\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$

23. $\displaystyle\lim_{x \to a} x = a$

24. $\displaystyle\lim_{x \to a} c = c$

25. $\displaystyle\lim_{x \to 0} x^2 = 0$

26. $\displaystyle\lim_{x \to 0} x^3 = 0$

27. $\displaystyle\lim_{x \to 0} |x| = 0$

28. $\displaystyle\lim_{x \to -6^+} \sqrt[8]{6 + x} = 0$

29. $\displaystyle\lim_{x \to 2} (x^2 - 4x + 5) = 1$

30. $\displaystyle\lim_{x \to 2} (x^2 + 2x - 7) = 1$

31. $\lim_{x \to -2} (x^2 - 1) = 3$

32. $\lim_{x \to 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x \to 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.

34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x \to 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

CAS **35.** (a) For the limit $\lim_{x \to 1} (x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.

(b) By using a computer algebra system to solve the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.

(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\displaystyle\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

37. Prove that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

$\left[\textit{Hint: } \text{Use } \left|\sqrt{x} - \sqrt{a}\right| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}}.\right]$

38. If H is the Heaviside function defined in Example 6 in Section 2.2, prove, using Definition 2, that $\lim_{t \to 0} H(t)$ does not exist. [*Hint:* Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1 in Section 2.3.

41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10{,}000$$

42. Prove, using Definition 6, that $\displaystyle\lim_{x \to -3} \frac{1}{(x + 3)^4} = \infty$.

43. Prove that $\lim_{x \to 0^+} \ln x = -\infty$.

44. Suppose that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = c$, where c is a real number. Prove each statement.

(a) $\lim_{x \to a} [f(x) + g(x)] = \infty$

(b) $\lim_{x \to a} [f(x)g(x)] = \infty$ if $c > 0$

(c) $\lim_{x \to a} [f(x)g(x)] = -\infty$ if $c < 0$

2.5 Continuity

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

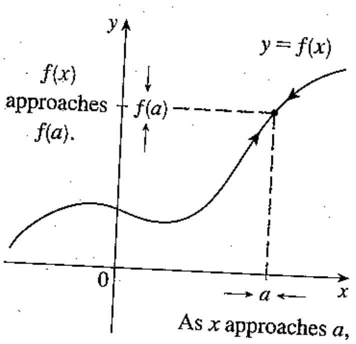


FIGURE 1

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a

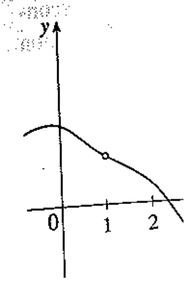


FIGURE 2

9. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.
(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

10. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

11. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x\to 2}[3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

12–14 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

12. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}, \quad a = 2$

13. $f(x) = (x + 2x^3)^4, \quad a = -1$

14. $h(t) = \dfrac{2t - 3t^2}{1 + t^3}, \quad a = 1$

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

15. $f(x) = \dfrac{2x+3}{x-2}, \quad (2, \infty)$

16. $g(x) = 2\sqrt{3-x}, \quad (-\infty, 3]$

17–22 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

17. $f(x) = \dfrac{1}{x+2} \qquad a = -2$

18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases} \qquad a = -2$

19. $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases} \qquad a = 0$

20. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \qquad a = 1$

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$

22. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \qquad a = 3$

23–24 How would you "remove the discontinuity" of f? In other words, how would you define $f(2)$ in order to make f continuous at 2?

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

25. $F(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$

26. $G(x) = \dfrac{x^2 + 1}{2x^2 - x - 1}$

27. $Q(x) = \dfrac{\sqrt[3]{x - 2}}{x^3 - 2}$

28. $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$

29. $A(t) = \arcsin(1 + 2t)$

30. $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$

31. $M(x) = \sqrt{1 + \dfrac{1}{x}}$

32. $N(r) = \tan^{-1}(1 + e^{-r^2})$

33–34 Locate the discontinuities of the function and illustrate by graphing.

33. $y = \dfrac{1}{1 + e^{1/x}}$

34. $y = \ln(\tan^2 x)$

35–38 Use continuity to evaluate the limit.

35. $\lim\limits_{x\to 4} \dfrac{5 + \sqrt{x}}{\sqrt{5 + x}}$

36. $\lim\limits_{x\to \pi} \sin(x + \sin x)$

37. $\lim\limits_{x\to 1} e^{x^2 - x}$

38. $\lim\limits_{x\to 2} \arctan\left(\dfrac{x^2 - 4}{3x^2 - 6x}\right)$

39–40 Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

41–43 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

41. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

42. $f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x - 3} & \text{if } x \geq 3 \end{cases}$

43. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

44. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

45. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

end HW7

46. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

HW8

47. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.
(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$
(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$
(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

48. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

49. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

50. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

51–54 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

51. $x^4 + x - 3 = 0, \quad (1, 2)$

52. $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

53. $e^x = 3 - 2x, \quad (0, 1)$

54. $\sin x = x^2 - x, \quad (1, 2)$

55–56 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

55. $\cos x = x^3$

56. $\ln x = 3 - 2x$

57–58 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

57. $100e^{-x/100} = 0.01x^2$

58. $\arctan x = 1 - x$

59. Prove that f is continuous at a if and only if

$$\lim_{h\to 0} f(a + h) = f(a)$$

60. To prove that sine is continuous, we need to show that $\lim_{x\to a} \sin x = \sin a$ for every real number a. By Exercise 59 an equivalent statement is that

$$\lim_{h\to 0} \sin(a + h) = \sin a$$

Use $\boxed{6}$ to show that this is true.

61. Prove that cosine is a continuous function.

62. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

63. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

64. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

65. Is there a number that is exactly 1 more than its cube?

66. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.