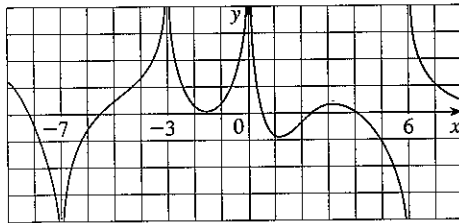


9. For the function f whose graph is shown, state the following.

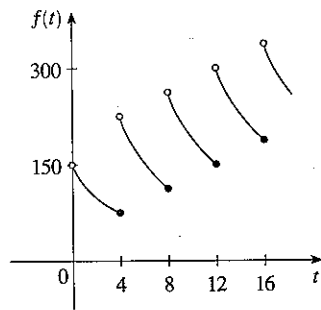
- (a) $\lim_{x \rightarrow -7} f(x)$ (b) $\lim_{x \rightarrow -3} f(x)$ (c) $\lim_{x \rightarrow 0} f(x)$
 (d) $\lim_{x \rightarrow 6^-} f(x)$ (e) $\lim_{x \rightarrow 6^+} f(x)$
 (f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \rightarrow 12^-} f(t) \quad \text{and} \quad \lim_{t \rightarrow 12^+} f(t)$$

and explain the significance of these one-sided limits.



11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \rightarrow a} f(x)$ exists.

11. $f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$

12. $f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leq x \leq \pi \\ \sin x & \text{if } x > \pi \end{cases}$

13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

- (a) $\lim_{x \rightarrow 0^-} f(x)$ (b) $\lim_{x \rightarrow 0^+} f(x)$ (c) $\lim_{x \rightarrow 0} f(x)$

13. $f(x) = \frac{1}{1 + e^{1/x}}$

14. $f(x) = \frac{x^2 + x}{\sqrt{x^3 + x^2}}$

End HWG

15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

15. $\lim_{x \rightarrow 0^-} f(x) = -1$, $\lim_{x \rightarrow 0^+} f(x) = 2$, $f(0) = 1$

16. $\lim_{x \rightarrow 0} f(x) = 1$, $\lim_{x \rightarrow 3^-} f(x) = -2$, $\lim_{x \rightarrow 3^+} f(x) = 2$,

HWG $f(0) = -1$, $f(3) = 1$

17. $\lim_{x \rightarrow 3^+} f(x) = 4$, $\lim_{x \rightarrow 3^-} f(x) = 2$, $\lim_{x \rightarrow -2} f(x) = 2$,
 $f(3) = 3$, $f(-2) = 1$

18. $\lim_{x \rightarrow 0^-} f(x) = 2$, $\lim_{x \rightarrow 0^+} f(x) = 0$, $\lim_{x \rightarrow 4^-} f(x) = 3$,
 $\lim_{x \rightarrow 4^+} f(x) = 0$, $f(0) = 2$, $f(4) = 1$

19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19. $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - x - 2}$,
 $x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$
 $1.9, 1.95, 1.99, 1.995, 1.999$

20. $\lim_{x \rightarrow -1} \frac{x^2 - 2x}{x^2 - x - 2}$,
 $x = 0, -0.5, -0.9, -0.95, -0.99, -0.999,$
 $-2, -1.5, -1.1, -1.01, -1.001$

21. $\lim_{t \rightarrow 0} \frac{e^{5t} - 1}{t}$, $t = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

22. $\lim_{h \rightarrow 0} \frac{(2+h)^5 - 32}{h}$,
 $h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

23–26 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}$

24. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\tan 5x}$

25. $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x^{10} - 1}$

26. $\lim_{x \rightarrow 0} \frac{9^x - 5^x}{x}$

27. (a) By graphing the function $f(x) = (\cos 2x - \cos x)/x^2$ and zooming in toward the point where the graph crosses the y -axis, estimate the value of $\lim_{x \rightarrow 0} f(x)$.
 (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

28. (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x}$$

by graphing the function $f(x) = (\sin x)/(\sin \pi x)$. State your answer correct to two decimal places.

- (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

29–37 Determine the infinite limit.

29. $\lim_{x \rightarrow -3^+} \frac{x+2}{x+3}$

30. $\lim_{x \rightarrow -3^-} \frac{x+2}{x+3}$

31. $\lim_{x \rightarrow 1} \frac{2-x}{(x-1)^2}$

32. $\lim_{x \rightarrow 5^-} \frac{e^x}{(x-5)^3}$

33. $\lim_{x \rightarrow 3^+} \ln(x^2 - 9)$

34. $\lim_{x \rightarrow \pi^-} \cot x$

35. $\lim_{x \rightarrow 2\pi^-} x \csc x$

36. $\lim_{x \rightarrow 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$

37. $\lim_{x \rightarrow 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6}$

38. (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

- (b) Confirm your answer to part (a) by graphing the function.

39. Determine $\lim_{x \rightarrow 1^-} \frac{1}{x^3 - 1}$ and $\lim_{x \rightarrow 1^+} \frac{1}{x^3 - 1}$

- (a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
 (b) by reasoning as in Example 9, and
 (c) from a graph of f .

40. (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y -axis, estimate the value of $\lim_{x \rightarrow 0} f(x)$.
 (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

41. (a) Estimate the value of the limit $\lim_{x \rightarrow 0} (1+x)^{1/x}$ to five decimal places. Does this number look familiar?
 (b) Illustrate part (a) by graphing the function $y = (1+x)^{1/x}$.

42. (a) Graph the function $f(x) = e^x + \ln|x-4|$ for $0 \leq x \leq 5$. Do you think the graph is an accurate representation of f ?
 (b) How would you get a graph that represents f better?

43. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 0.8, 0.6, 0.4, 0.2, 0.1,$ and 0.05 , and guess the value of

$$\lim_{x \rightarrow 0} \left(x^2 - \frac{2^x}{1000} \right)$$

- (b) Evaluate $f(x)$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003,$ and 0.001 . Guess again.

44. (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1, 0.5, 0.1, 0.05, 0.01,$ and 0.005 .

- (b) Guess the value of $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$.

- (c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.4 a method for evaluating the limit will be explained.)
 (d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y -axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h . Compare with the results of part (c).

45. Graph the function $f(x) = \sin(\pi/x)$ of Example 4 in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Then zoom in toward the origin several times. Comment on the behavior of this function.

46. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \rightarrow c^-$?

47. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \quad -\pi \leq x \leq \pi$$

Then find the exact equations of these asymptotes.

48. (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

- (b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

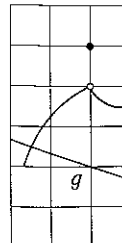


FIGURE 1

approach 0. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1 , we can write

$$-1 \leq \sin \frac{1}{x} \leq 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geq 0$ for all x and so, multiplying each side of the inequalities in [4] by x^2 , we get

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

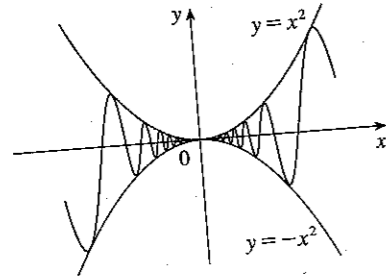


FIGURE 8
 $y = x^2 \sin(1/x)$

2.3 Exercises

1. Given that

$$\lim_{x \rightarrow 2} f(x) = 4 \quad \lim_{x \rightarrow 2} g(x) = -2 \quad \lim_{x \rightarrow 2} h(x) = 0$$

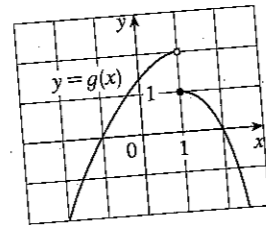
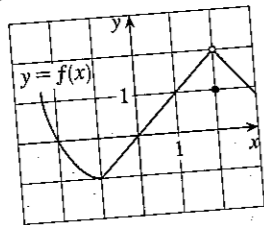
find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \rightarrow 2} [f(x) + 5g(x)]$ (b) $\lim_{x \rightarrow 2} [g(x)]^3$

(c) $\lim_{x \rightarrow 2} \sqrt{f(x)}$ (d) $\lim_{x \rightarrow 2} \frac{3f(x)}{g(x)}$

(e) $\lim_{x \rightarrow 2} \frac{g(x)}{h(x)}$ (f) $\lim_{x \rightarrow 2} \frac{g(x)h(x)}{f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(a) $\lim_{x \rightarrow 2} [f(x) + g(x)]$

(b) $\lim_{x \rightarrow 1} [f(x) + g(x)]$

(c) $\lim_{x \rightarrow 0} [f(x)g(x)]$

(d) $\lim_{x \rightarrow 1} \frac{f(x)}{g(x)}$

(e) $\lim_{x \rightarrow 2} [x^3 f(x)]$

(f) $\lim_{x \rightarrow 1} \sqrt{3 + f(x)}$

3-9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \rightarrow 3} (5x^3 - 3x^2 + x - 6)$

4. $\lim_{x \rightarrow -1} (x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t \rightarrow 2} \frac{t^4 - 2}{2t^2 - 3t + 2}$ 6. $\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$

7. $\lim_{x \rightarrow 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$ 8. $\lim_{t \rightarrow 2} \left(\frac{t^2 - 2}{t^3 - 3t + 5} \right)^2$

9. $\lim_{x \rightarrow 2} \sqrt{\frac{2x^2 + 1}{3x - 2}}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} (x + 3)$$

is correct.

11-32 Evaluate the limit, if it exists.

11. $\lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5}$

12. $\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$

13. $\lim_{x \rightarrow 5} \frac{x^2 - 5x + 6}{x - 5}$

14. $\lim_{x \rightarrow -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$

15. $\lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim_{x \rightarrow 1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

17. $\lim_{h \rightarrow 0} \frac{(-5 + h)^2 - 25}{h}$

18. $\lim_{h \rightarrow 0} \frac{(2 + h)^3 - 8}{h}$

19. $\lim_{x \rightarrow -2} \frac{x + 2}{x^3 + 8}$

20. $\lim_{t \rightarrow 1} \frac{t^4 - 1}{t^3 - 1}$

21. $\lim_{h \rightarrow 0} \frac{\sqrt{9 + h} - 3}{h}$

22. $\lim_{u \rightarrow 2} \frac{\sqrt{4u + 1} - 3}{u - 2}$

23. $\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

24. $\lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x^4 - 1}$

25. $\lim_{t \rightarrow 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$

26. $\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

27. $\lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{16x - x^2}$

28. $\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$

29. $\lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$

30. $\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$

31. $\lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h}$

32. $\lim_{h \rightarrow 0} \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$

33. (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

- (b) by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.
 (c) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.
 (d) Use the Limit Laws to prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

- to estimate the value of $\lim_{x \rightarrow 0} f(x)$ to two decimal places.
 (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
 (c) Use the Limit Laws to find the exact value of the limit.

35. Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} (x^2 \cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

36. Use the Squeeze Theorem to show that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f , g , and h (in the notation of the Squeeze Theorem) on the same screen.

37. If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \rightarrow 4} f(x)$.

38. If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x , evaluate $\lim_{x \rightarrow 1} g(x)$.

39. Prove that $\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x} = 0$.

40. Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0$.

41-46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim_{x \rightarrow 3} (2x + |x - 3|)$

42. $\lim_{x \rightarrow -6} \frac{2x + 12}{|x + 6|}$

43. $\lim_{x \rightarrow 0.5^-} \frac{2x - 1}{|2x^3 - x^2|}$

44. $\lim_{x \rightarrow -2} \frac{2 - |x|}{2 + x}$

45. $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

46. $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

47. The signum (or sign) function, denoted by sgn , is defined by

$$\text{sgn } x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- (a) Sketch the graph of this function.
 (b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \rightarrow 0^+} \text{sgn } x$ (ii) $\lim_{x \rightarrow 0^-} \text{sgn } x$
 (iii) $\lim_{x \rightarrow 0} \text{sgn } x$ (iv) $\lim_{x \rightarrow 0} |\text{sgn } x|$

48. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geq 1 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.
 (b) Does $\lim_{x \rightarrow 1} f(x)$ exist?
 (c) Sketch the graph of f .

49. Let $g(x) = \frac{x^2 + x - 6}{|x - 2|}$.

- (a) Find
 (i) $\lim_{x \rightarrow 2^+} g(x)$ (ii) $\lim_{x \rightarrow 2^-} g(x)$
 (b) Does $\lim_{x \rightarrow 2} g(x)$ exist?
 (c) Sketch the graph of g .

50. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

- (i) $\lim_{x \rightarrow 1^-} g(x)$ (ii) $\lim_{x \rightarrow 1} g(x)$ (iii) $g(1)$
 (iv) $\lim_{x \rightarrow 2^-} g(x)$ (v) $\lim_{x \rightarrow 2^+} g(x)$ (vi) $\lim_{x \rightarrow 2} g(x)$

(b) Sketch the graph of g .

51. (a) If the symbol $\llbracket \cdot \rrbracket$ denotes the greatest integer function defined in Example 10, evaluate

- (i) $\lim_{x \rightarrow -2^+} \llbracket x \rrbracket$ (ii) $\lim_{x \rightarrow -2} \llbracket x \rrbracket$ (iii) $\lim_{x \rightarrow -2.4} \llbracket x \rrbracket$

(b) If n is an integer, evaluate

- (i) $\lim_{x \rightarrow n^-} \llbracket x \rrbracket$ (ii) $\lim_{x \rightarrow n^+} \llbracket x \rrbracket$

(c) For what values of a does $\lim_{x \rightarrow a} \llbracket x \rrbracket$ exist?

52. Let $f(x) = \llbracket \cos x \rrbracket$, $-\pi \leq x \leq \pi$.

(a) Sketch the graph of f .

(b) Evaluate each limit, if it exists.

- (i) $\lim_{x \rightarrow 0} f(x)$ (ii) $\lim_{x \rightarrow (\pi/2)^-} f(x)$
 (iii) $\lim_{x \rightarrow (\pi/2)^+} f(x)$ (iv) $\lim_{x \rightarrow \pi/2} f(x)$

(c) For what values of a does $\lim_{x \rightarrow a} f(x)$ exist?

53. If $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$, show that $\lim_{x \rightarrow 2} f(x)$ exists but is not equal to $f(2)$.

54. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \rightarrow c^-} L$ and interpret the result. Why is a left-hand limit necessary?

55. If p is a polynomial, show that $\lim_{x \rightarrow a} p(x) = p(a)$.

56. If r is a rational function, use Exercise 55 to show that $\lim_{x \rightarrow a} r(x) = r(a)$ for every number a in the domain of r .

57. If $\lim_{x \rightarrow 1} \frac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \rightarrow 1} f(x)$.

58. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$, find the following limits.

- (a) $\lim_{x \rightarrow 0} f(x)$ (b) $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

59. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x) = 0$.

60. Show by means of an example that $\lim_{x \rightarrow a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists.

61. Show by means of an example that $\lim_{x \rightarrow a} [f(x)g(x)]$ may exist even though neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists.

62. Evaluate $\lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$.

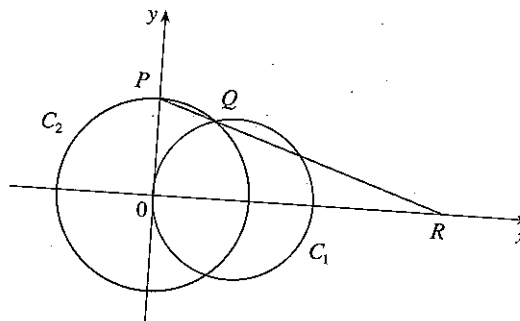
End HWB

63. Is there a number a such that

$$\lim_{x \rightarrow -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

64. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x -axis. What happens to R as C_2 shrinks, that is, as $r \rightarrow 0^+$?



2.4 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes because such phrases as “ x is close to 2” and “ $f(x)$ gets closer and closer to L ” are vague. In order to be able to prove conclusively that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0.0001 \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.