2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30.
(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.
(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.
(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$
What are your conclusions?

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.
(a) If Q is the point $(x, 1/(1 - x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.
(a) If Q is the point $(x, \cos \pi x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 0 (ii) 0.4 (iii) 0.49 (iv) 0.499
(v) 1 (vi) 0.6 (vii) 0.51 (viii) 0.501
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.
(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.
(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 0.5 second (ii) 0.1 second
(iii) 0.05 second (iv) 0.01 second
(b) Estimate the instantaneous velocity when $t = 2$.

6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.
(a) Find the average velocity over the given time intervals:
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]
(b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a cyclist.

t (seconds)	0	1	2	3	4	5
s (meters)	0	1.4	5.1	10.7	17.7	25.8

(a) Find the average velocity for each time period:
(i) [1, 3] (ii) [2, 3] (iii) [3, 5] (iv) [3, 4]
(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.
(a) Find the average velocity during each time period:
(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]
(b) Estimate the instantaneous velocity of the particle when $t = 1$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9. Do the slopes appear to be approaching a limit?
(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.
(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

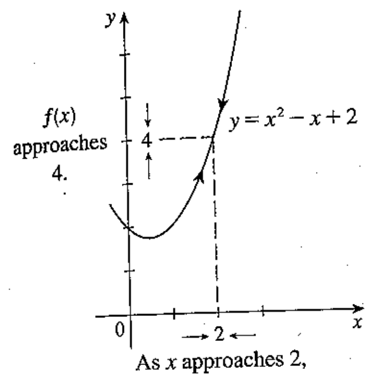


FIGURE 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

$\boxed{1}$ **Definition** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. (A more precise definition will be given in Section 2.4.)

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5$, 10, 20, 25, and 30.
(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.
(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.
(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$
What are your conclusions?

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.
(a) If Q is the point $(x, 1/(1 - x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.
(a) If Q is the point $(x, \cos \pi x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 0 (ii) 0.4 (iii) 0.49 (iv) 0.499
(v) 1 (vi) 0.6 (vii) 0.51 (viii) 0.501
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.
(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.
(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 0.5 second (ii) 0.1 second
(iii) 0.05 second (iv) 0.01 second
(b) Estimate the instantaneous velocity when $t = 2$.

6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.
(a) Find the average velocity over the given time intervals:
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]
(b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a cyclist.

t (seconds)	0	1	2	3	4	5
s (meters)	0	1.4	5.1	10.7	17.7	25.8

(a) Find the average velocity for each time period:
(i) [1, 3] (ii) [2, 3] (iii) [3, 5] (iv) [3, 4]
(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.
(a) Find the average velocity during each time period:
(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]
(b) Estimate the instantaneous velocity of the particle when $t = 1$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2$, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8, and 0.9. Do the slopes appear to be approaching a limit?
(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.
(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

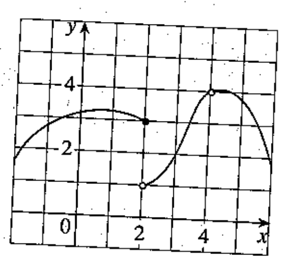
3. Explain the meaning of each of the following.

(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$

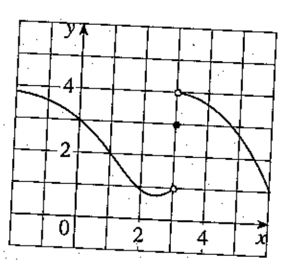
(d) $f(2)$ (e) $\lim_{x \to 4} f(x)$ (f) $f(4)$



5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$



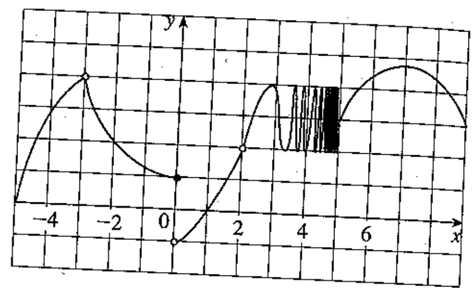
6. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$

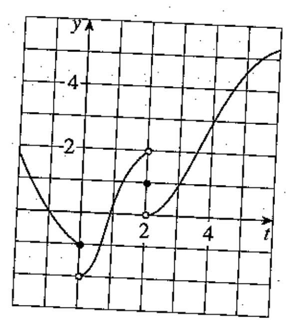


7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$

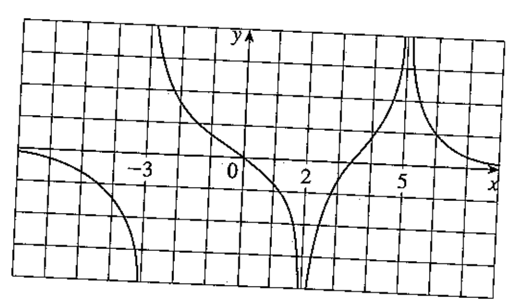


8. For the function R whose graph is shown, state the following.

(a) $\lim_{x \to 2} R(x)$ (b) $\lim_{x \to 5} R(x)$

(c) $\lim_{x \to -3^-} R(x)$ (d) $\lim_{x \to -3^+} R(x)$

(e) The equations of the vertical asymptotes.

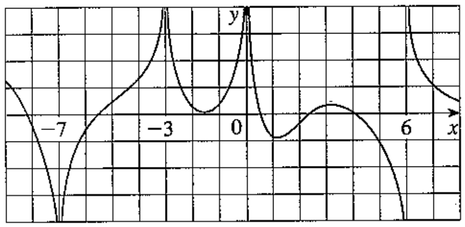


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

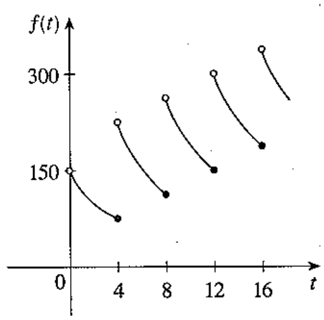
(f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \to 12^-} f(t) \qquad \text{and} \qquad \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

11. $f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leqslant x < 1 \\ 2 - x & \text{if } x \geqslant 1 \end{cases}$

12. $f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leqslant x \leqslant \pi \\ \sin x & \text{if } x > \pi \end{cases}$

13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

13. $f(x) = \dfrac{1}{1 + e^{1/x}}$ **14.** $f(x) = \dfrac{x^2 + x}{\sqrt{x^3 + x^2}}$

15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

15. $\lim_{x \to 0^-} f(x) = -1, \quad \lim_{x \to 0^+} f(x) = 2, \quad f(0) = 1$

16. $\lim_{x \to 0} f(x) = 1, \quad \lim_{x \to 3^-} f(x) = -2, \quad \lim_{x \to 3^+} f(x) = 2,$
$f(0) = -1, \quad f(3) = 1$

17. $\lim_{x \to 3^+} f(x) = 4, \quad \lim_{x \to 3^-} f(x) = 2, \quad \lim_{x \to -2} f(x) = 2,$
$f(3) = 3, \quad f(-2) = 1$

18. $\lim_{x \to 0^-} f(x) = 2, \quad \lim_{x \to 0^+} f(x) = 0, \quad \lim_{x \to 4^-} f(x) = 3,$
$\lim_{x \to 4^+} f(x) = 0, \quad f(0) = 2, \quad f(4) = 1$

19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19. $\displaystyle\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2},$
$x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$
$1.9, 1.95, 1.99, 1.995, 1.999$

20. $\displaystyle\lim_{x \to -1} \frac{x^2 - 2x}{x^2 - x - 2},$
$x = 0, -0.5, -0.9, -0.95, -0.99, -0.999,$
$-2, -1.5, -1.1, -1.01, -1.001$

21. $\displaystyle\lim_{t \to 0} \frac{e^{5t} - 1}{t}, \quad t = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

22. $\displaystyle\lim_{h \to 0} \frac{(2 + h)^5 - 32}{h},$
$h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

23–26 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x}$ **24.** $\displaystyle\lim_{x \to 0} \frac{\tan 3x}{\tan 5x}$

25. $\displaystyle\lim_{x \to 1} \frac{x^6 - 1}{x^{10} - 1}$ **26.** $\displaystyle\lim_{x \to 0} \frac{9^x - 5^x}{x}$

27. (a) By graphing the function $f(x) = (\cos 2x - \cos x)/x^2$ and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.
(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.