Summary of Proofs for Exponents and Logs

We will take the following facts as given:

1. For $n > 0$ an integer, $e^n = e \cdot e \cdot \ldots \cdot e$, a product of n copies of e.
2. $e^0 = 1$.
3. $e > 1$.
4. $e^{x+y} = e^x \cdot e^y$ for all x and y.
5. $e^{-x} = 1/e^x$ for all x.
6. $(e^x)^y = e^{xy}$ for all x and y.
7. For a and b integers with $b > 0$, $e^{a/b} = \sqrt[b]{e^a}$.
8. For $n > 0$ an integer, $\sqrt[n]{x}$ is an increasing function.

1 Exponential Functions

1.1 e^x is an Increasing Function

Our first goal is to show that e^x is an increasing function. We cannot do this all at once, and so must break the proof into several steps. First, we will show that the claim holds for positive integers, then for all integers, then for all rational numbers.

1. *For m and n integers with $m > n \geq 0$, we have $e^m > e^n$.*

   Proof: Since $m > n$, e^m is a product involving more copies of e than e^n. Since $e > 1$, multiplying more copies of e gives us a bigger number. Therefore $e^m > e^n$.

2. *For m and n integers with $m > n$, we have $e^m > e^n$.*

   Proof: We know that this works for non-negative numbers, so subtract n from both sides in order to make both sides non-negative (this may be slightly non-intuitive, but allows us to prove the two cases $0 > m > n$ and $m > 0 > n$ at the same time).

$$\begin{aligned} m &> n \\ m - n &> 0 \\ e^{m-n} &> e^0 \\ e^m/e^n &> 1 \\ e^m &> e^n \end{aligned}$$

3. *For $a/b > c/d$, with $b, d > 0$, we have $e^{a/b} > e^{c/d}$.*

   Proof: We know that $e^m > e^n$ if $m > n$ and m, n are integers, so clear the denominators in order to get an inequality involving integers. Once we have this, we can take roots to get back to fractional exponents.

$$\begin{aligned} a/b &> c/d \\ ad &> bc \\ e^{ad} &> e^{bc} \\ (e^{ad})^{1/bd} &> (e^{bc})^{1/bd} \\ e^{a/b} &> e^{c/d} \end{aligned}$$

   Note that the second-to-last step uses the fact that $\sqrt[bd]{x}$ is an increasing function.

Therefore e^x is increasing on the rational numbers. We have not yet defined e^r for irrational numbers r, but we now define e^r so that e^x is increasing on the real numbers. As the textbook mentions, this would imply that

$$e^{3.1} < e^{3.14} < \ldots < e^{\pi} < \ldots < e^{3.142} < e^{3.15}$$

This gives us our first important result:

Theorem 1.1 *e^x is an increasing function.*

1.2 Other Results for e^x

Later in the course, we will get a much more powerful tool for proving that certain functions are increasing:

Theorem 1.2 *If f is differentiable and $f'(x) > 0$ for all x, then f is increasing.*

But for now we move on to another important theorem, which will allow us to say more about our function e^x:

Theorem 1.3 *If f is increasing, then f is 1-1.*

Proof: Let $x \neq y$, so either $x > y$ or $x < y$.
If $x > y$, then $f(x) > f(y)$, so $f(x) \neq f(y)$.
If $x < y$, then $f(x) < f(y)$, so again we get $f(x) \neq f(y)$. Therefore $x \neq y \Rightarrow f(x) \neq f(y)$, and so f is 1-1.

Side note: Since the names for x and y were arbitrary, we did not actually need to prove the cases $x > y$ and $x < y$ separately. It would have been okay to say "without loss of generality (WLOG), let $x > y$." We can also prove a similar result for decreasing functions (where $x > y \Rightarrow f(x) < f(y)$):

Corollary 1.3.1 *If f is decreasing, then f is 1-1.*

Next, we can put two of our theorems together to conclude something new about e^x:

Corollary 1.3.2 *e^x is 1-1.*

Proof: e^x is increasing, and increasing functions are 1-1.

1.3 Other Bases

There is no particular reason that these results should work for e^x only, so what happens when we look at a^x for other values of a? If $a < 0$, then a^x does not behave nicely as a real-valued function, so we will ignore it. If $a = 1$, then $a^x = 1$ for all x. But if $a > 0$ and $a \neq 1$, then we can prove similar results about a^x by using the same methods as we did for e^x.

If $a > 1$ then by simply substituting a for e everywhere, we get the following:

Theorem 1.4 *If $a > 1$, then a^x is an increasing function.*

If $a < 1$, then a^n will get smaller rather than larger as n increases. By reversing the inequalities in the proof for e^x, we get:

Theorem 1.5 *If $0 < a < 1$, then a^x is a decreasing function.*

Either way, we can apply our theorems about increasing and decreasing functions being 1-1 and conclude that

Corollary 1.5.1 *For $a > 0$, $a \neq 1$, a^x is 1-1. Each such function therefore has an inverse function, which we will call $\log_a$ (or $\ln$, in the case $a = e$).*

2 Logarithmic Functions

The key idea that Ole wished to drive across is that the rules for evaluating logarithms do not exist in a vacuum, but are immediate consequences of the fact that logarithmic functions are inverses of exponential functions. With this in mind, we will set out to prove the log laws based on what we now know about exponentials.

2.1 Basic Relations

Since the range of a^x is $(0, \infty)$ for any base $a > 0, a \neq 1$, the domain of $\log_a$ is also $(0, \infty)$. By definition, $\log_a(a^x) = x$ for all x. Also by definition, $a^{\log_a(x)} = x$ for all $x > 0$.

For the purpose of proofs, it will sometimes be easier to say $\log_a(x) = c \Leftrightarrow a^c = x$. We start by proving one method for converting between bases:

Theorem 2.1 $a^x = b^{x \log_b(a)}$

Proof: $a^x = (b^{\log_b a})^x = b^{x \log_b a}$.

In particular, if we want to make e our default base this implies that $a^x = e^{x \ln(a)}$. This can be handy for computations because calculators tend to have e and 10 as their default bases for taking exponents and logarithms.

2.2 The Log Laws

For several of the folowing proofs we will make temporary substitutions by setting $\log_a(x)$ to some constant c, then saying $a^c = x$. This is not strictly necessary for the proof, but will hopefully make the notation easier to follow.

Theorem 2.2 $\log_a(1) = 0; \log_a(a) = 1$

Proof: $a^0 = 1; a^1 = a$.

Theorem 2.3 $\log_a(xy) = \log_a(x) + \log_a(y)$

Proof: Set $b = \log_a(xy), c = \log_a(x), d = \log_a(y)$, so that we have $a^b = xy, a^c = x, a^d = y$. We then get

$$\begin{aligned} a^b &= xy \\ a^b &= a^c a^d \\ a^b &= a^{c+d} \\ b &= c+d \\ \log_a(xy) &= \log_a(x) + \log_a(y) \end{aligned}$$

Theorem 2.4 $\log_a(x/y) = \log_a(x) - \log_a(y)$

Proof: Set $b = \log_a(x/y), c = \log_a(x), d = \log_a(y)$, so that $a^b = x/y, a^c = x, a^d = y$. We then get

$$\begin{aligned} a^b &= x/y \\ a^b &= a^c/a^d \\ a^b &= a^{c-d} \\ b &= c-d \\ \log_a(x/y) &= \log_a(x) - \log_a(y) \end{aligned}$$

Theorem 2.5 $\log_a(x^r) = r\log_a(x)$

Proof:
Set $b = \log_a(x^r), c = \log_a(x)$, so that $a^b = x^r, a^c = x$. We then get

$$\begin{aligned} a^b &= x^r \\ a^b &= (a^c)^r \\ b &= rc \\ \log_a(x^r) &= r\log_a(x) \end{aligned}$$

This final result gives us a method for converting between bases:

Theorem 2.6 *For any bases a and b, we have* $\log_a(x) = \log_b(x)/\log_b(a)$

Proof: Let $c = \log_a(x)$, so that $a^c = x$. We then get

$$\begin{aligned} a^c &= x \\ \log_b(a^c) &= \log_b(x) \\ c\log_b(a) &= \log_b(x) \\ c &= \log_b(x)/\log_b(a) \\ \log_a(x) &= \log_b(x)/\log_b(a) \end{aligned}$$

If we take $b = e$ in particular, this gives us $\log_a(x) = \ln(x)/\ln(a)$.