Solutions to Even Problems, Section 4.2

16) $f(x) = \begin{cases} 2-(2x-1) & 2x-1 \geq 0 \\ 2+2x-1 & 2x-1 < 0 \end{cases}$, so $f'(x) = \begin{cases} -2 & 2x-1 > 0 \\ 2 & 2x-1 < 0 \end{cases}$. But $\frac{f(3)-f(0)}{3-0} = -4/3$, which is not equal to $f'(c)$ for any c. The Mean Value Theorem does not apply because f is not differentiable at $x = 1/2$.

18) $f(-1) = -1 + 1/e < 0$ and $f(0) = 1 > 0$, so (since f is continuous) the Intermediate Value Theorem implies that there exists a $c \in (-1, 0)$ so that $f(c) = 0$. Then not that $f'(x) = 3x^2 + e^x \geq e^x > 0$ for all x. Thus f is an increasing function, so $f(y) < 0$ for $y < c$ and $f(y) > 0$ for $y > c$, and so c is the only zero.

Alternately, if f had two zeros, then Rolle's theorem would imply that f' had a zero somewhere inbetween, which contradicts the fact that $f'(x) > 0$.

20) $f'(x) = 4x^3 + 4$, which has at most one real root since x^3 is an increasing function (and so $4x^3 + 4$ is also increasing). Thus $f(x)$ has at most two real roots.

(Longer: If f had three roots, then Rolle's Theorem would imply that f' had at least two roots, which it doesn't.)

22)

a) Rolle's Theorem.

b) By Rolle's Theorem, $f'(x)$ has at least two real roots. Since $f(x)$ is twice differentiable, we know that $f'(x)$ is a differentiable function. Since $f'(x)$ is differentiable and has two real roots, applying Rolle's theorem to $f'(x)$ tells us that its derivative (which is $f''(x)$) has at least one real root.

c) If f is n times differentiable on $\mathbb{R}$ and has $(n+1)$ real roots, then $f^{(n)}$ has at least one real root.

24) By the MVT, there exists $c \in (2, 8)$ so that $f'(c) = \frac{f(8)-f(2)}{8-2}$. Since $3 \leq f'(c) \leq 5$, we get that $3 \leq \frac{f(8)-f(2)}{6} \leq 5$, and so $18 \leq f(8) - f(2) \leq 30$.

26) Let $h = f - g$. Then by the MVT, there exists $c \in (a, b)$ such that $h'(c) = \frac{h(b)-h(a)}{b-a}$. But $h'(c) > 0$, and so we get $0 < \frac{f(b)-g(b)}{b-a}$. It follows that $f(b) - g(b) > 0$, and so $f(b) > g(b)$.

28) f is odd, so $f(-b) = -f(b)$. By the MVT, there exists a $c \in (-b, b)$ such that

$$f'(c) = \frac{f(b) - f(-b)}{b - (-b)} = \frac{2f(b)}{2b} = \frac{f(b)}{b}$$

30) Let $g(x) = cx$. Then $f'(x) = g'(x) = c$ for all x, so Corollary 7 implies that $f(x) = g(x) + d = cx + d$ for some constant d.

32) Take derivatives of both sides.

$$\frac{d}{dx} 2\arcsin(x) = \frac{2}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos(1-2x^2) = \frac{-1}{\sqrt{1-(1-2x^2)^2}}(-4x) = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

Thus by Corollary 7 the two functions are the same up to a constant. Since both are equal to zero when $x = 0$, that constant is zero, and so the two functions are equal.

34) Use the fact that acceleration is the derivative of velocity, and that 10 minutes is 1/6 of an hour. Then (assuming that the velocity of the car is a differentiable function) the MVT implies that there is some $t \in (2:00, 2:10)$ such that $f'(t) = (50 - 30mph)/(1/6h) = 120mi/h^2$.

36) Suppose $f(a) = a, f(b) = b$ for some $a, b \in \mathbb{R}$. Then the MVT implies that $f'(c) = \frac{f(a)-f(b)}{a-b} = \frac{a-b}{a-b} = 1$ for some $c \in (a, b)$. This contradicts the assertion that $f'(x) \neq 1$ for all real numbers x, so f has at most one number a such that $f(a) = a$.

Alternately, let $g(x) = f(x) - x$. We would then have that $g'(x) \neq 0$, and want to show that g has at most one real zero, which follows from Rolle's Theorem.