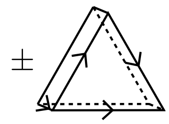
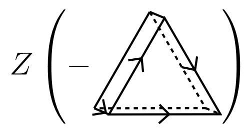
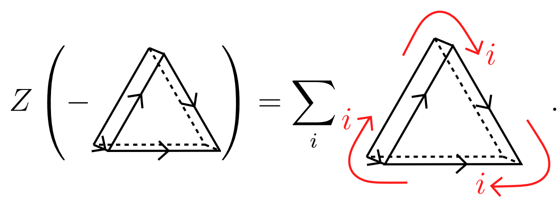
Let Z refer to the 3D TQFT and let $\{V_i\}$ will be a complete collection of simple $D = D(G)$-modules. Write



to denote $\pm(\Delta^2 \times S^1)$. The two triangle faces pictured are glued together. Then

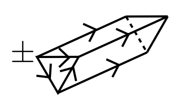


is the multiplication in D so that, like the 2D situation,

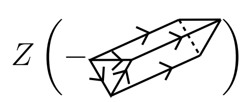


Observe that folding together two sides of $+\Delta^2 \times S^1$ produces a solid torus $D^2 \times S^1$ such that $Z(D^2 \times S^1)$ is the identity in D.

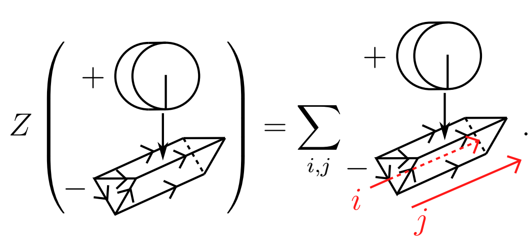
Similarly, let an elongated triangular prism



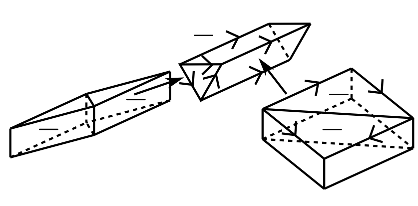
denote $\pm I \times \Delta^2$ with the three horizontal edges identified. In particular



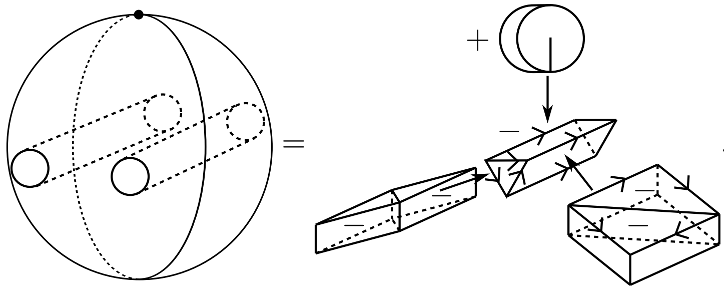
is the comultiplication in D. Note that gluing $D^2 \times S^1$ to one of side of this corresponds to plugging the identity into the comultiplication Δ. Hence



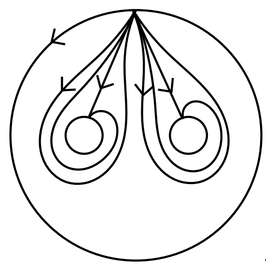
Note that



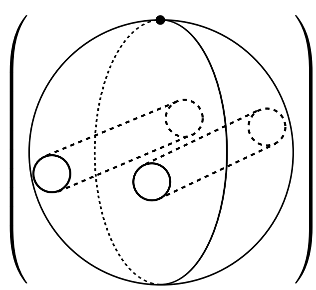
forms a pair of pants crossed with an interval. Therefore, after a gluing a solid torus on top:



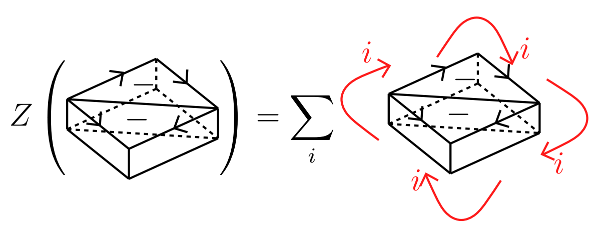
the triangulation on the front (not drawn) is



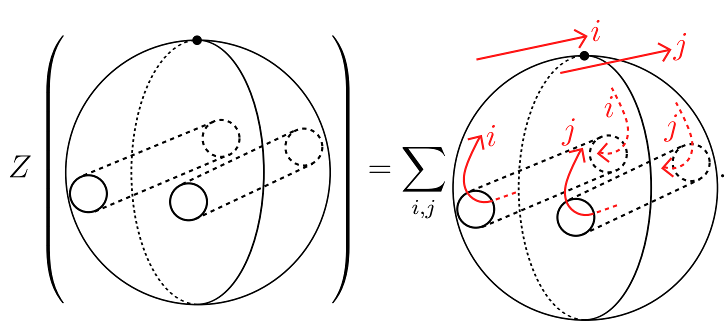
There is a similar triangulation on the other side. It should be noted that, after ungluing and forgetting the two triangles,

Z  $\in D^{\otimes 2} \otimes (D^*)^{\otimes 4}$.

Since

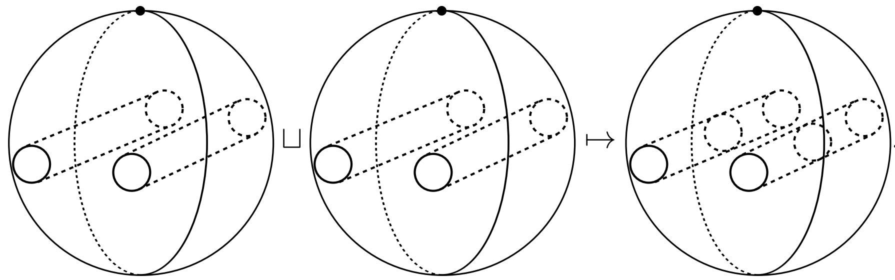


it follows that

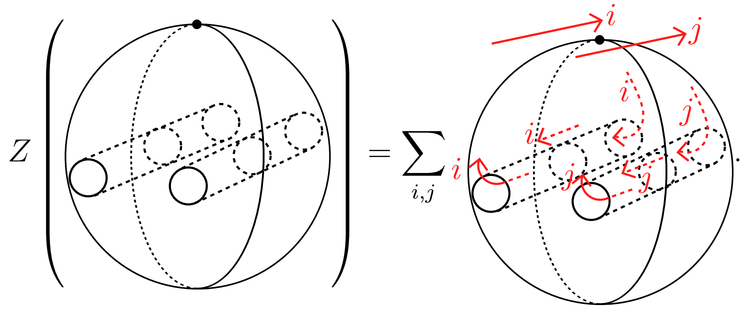


(1)

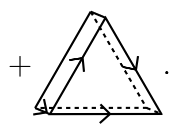
Note that gluing two of these together does not give quite the same Δ-complex structure on the boundary:



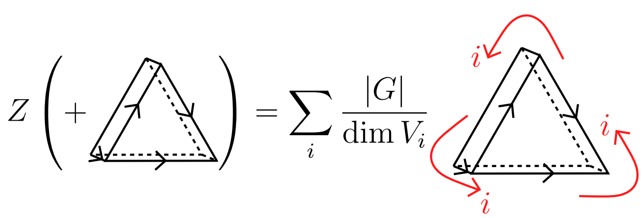
Using the gluing theorem, then



To get back to the original triangulation, glue on two copies of



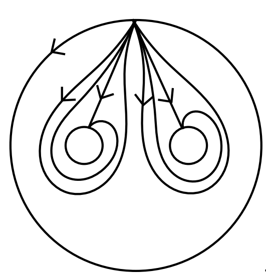
Since



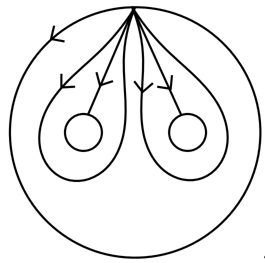
gluing on the two copies of $+\Delta^2 \times S^1$ introduces an extra factor of $\frac{|G|}{\dim V_i}\frac{|G|}{\dim V_j}$. This factor cancels with two new red loops and two new interior vertices.

Let L be a 2-manifold with Δ-complex structure. $\mathrm{Hom}(L, G)$ is defined to be $\mathbb{C}\,\mathrm{Hom}(\pi_1(|L|; L^0), G)$, which only depends on the vertices of L. Therefore if $|L| = |L'|$ and $L^0 = (L')^0$, then $\mathrm{Hom}(L, G)$ is exactly the same set as $\mathrm{Hom}(L', G)$.

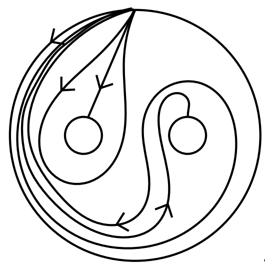
Specialize to the case where L is the following complex:



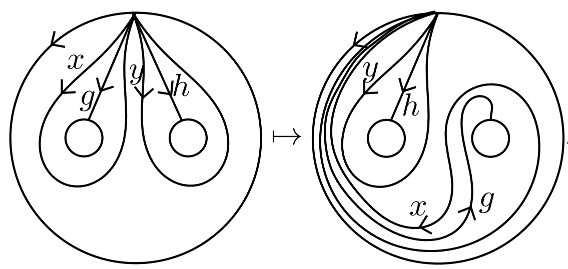
For simplicity, draw it without some of the edges:



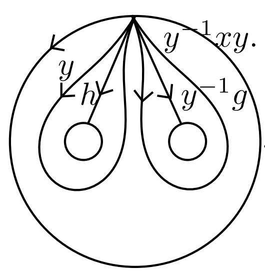
There's a homeomorphism of $|L|$ given by a half twist:



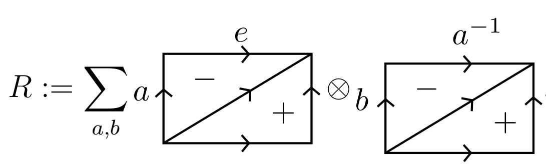
Call the resulting complex L'. The $\mathrm{Hom}(L,G) = \mathrm{Hom}(L',G)$ and precomposition with the (inverse of the) half twist provides a a map $\mathrm{Hom}(L,G) \to \mathrm{Hom}(L',G) = \mathrm{Hom}(L,G)$ given by



In terms of the original triangulation, the latter is

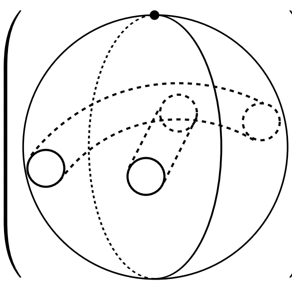
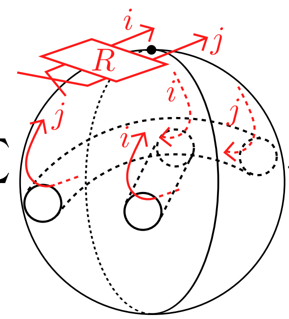


This triangulation identifies $\mathbb{C}\,\mathrm{Hom}(L,G)$ with $D \otimes D$. In terms of this identification, the half twist is given by switching the two factors in $D \otimes D$ and then left multiplying by



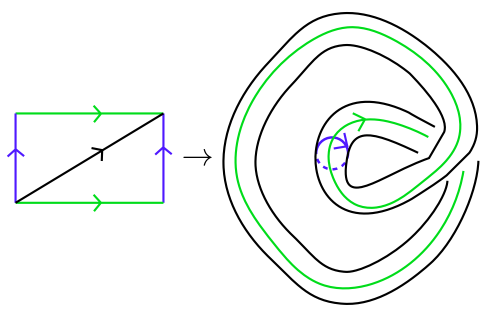
What is named R here is traditionally named R^{-1}, but which one you name R and which one you name R^{-1} is a convention choice. For historical reasons, R is called a "universal R-matrix."

Applying the half twist to the twice-gored ball gives

Z () $= \sum_{i,j}$ 

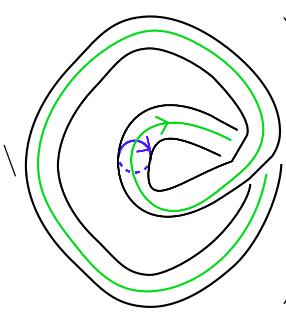
. (2)

Picking a meridian and longitude on a knot determines a Δ-complex structure on the boundary which can be unglued along the meridian to give an identification with a triangulation of a cylinder:

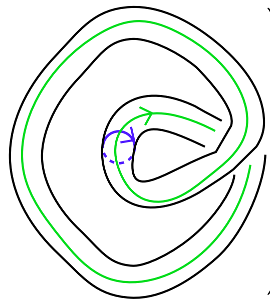


Such an identification is often called a "framing" of the knot. When the longitude runs parallel to the picture of the knot, as above, the framing is called the "blackboard framing."

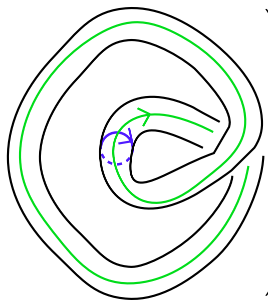
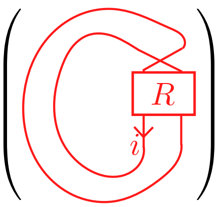
In the conventions of this lecture, the boundary always picks up the $-$ orientation on $I \times S^1$, so that after ungluing,

$Z\left(S^3 \setminus \right.$  $\left. \right) \in D^*.$

In fact it's in the center of D^* and so can be written

$Z\Big(S^3 \setminus$  $\Big) = \sum_i a_i \chi_i.$

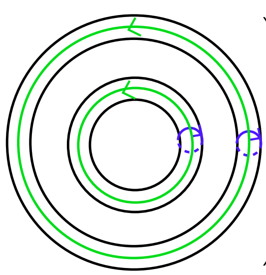
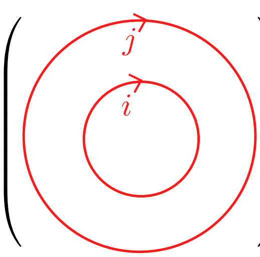
The coefficients can be determined simply by gluing together the two sides of (2). Hence

$Z\Big(S^3 \setminus$  $\Big) = \frac{1}{|G|}\sum_i \Big($ 

$\Big)\chi_i.$

The significant takeaway is that the coefficient a_i is expressed in terms of some tensor diagram. In this case, that tensor diagram gives the trace of R in $V_i \otimes V_i$ (times $\frac{1}{|G|}$). The factor of $\frac{1}{|G|}$ comes from one new interior vertex. The red tensor diagram is the Reshetikhin-Turaev invariant of the ribbon Hopf algebra D.

A similar example is the 0-framed unlink, which can be obtained by gluing together the two sides of (1):

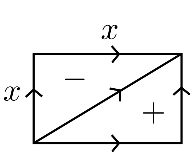
$Z\Big(S^3 \setminus$  $\Big) = \frac{1}{|G|}\sum_{i,j} \Big($ 

$\Big)\chi_i \otimes \chi_j.$

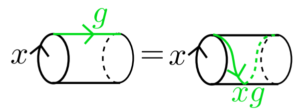
Of course the red arrows here evaluate to $\dim V_i \dim V_j$.

In general, for the *blackboard framing* the tensor diagram is obtained by taking the green longitude, making it red, reversing its direction, and putting an R at each crossing. This works for more complicated links that what are drawn here.

To handle twists in the green longitude, one must multiply suitably by the central element

$\sum_x x$ 

since



(equality as representations of the fundamental groupoid).