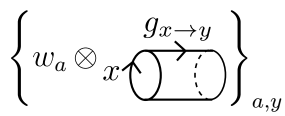
Let $D_x$ be the subalgebra of $D$ given by labelings with both meridians labeled by $x$. The longitude labeling gives a canonical isomorphism $D_x \cong \mathbb{C}C_x$.

Here is a construction of modules of $D$ from modules of $D_x$. If $W$ is a (right) $D_x$-module then

$$W \underset{D_x}{\otimes} D$$

is a right $D$-module. For each $y$ conjugate to $x$, fix $g_{x\to y} \in G$ conjugating $x$ to $y$. That is, $g_{x\to y}^{-1} x g_{x\to y} = y$. Let $\{w_a\}$ be a basis of $W$. Then




forms a basis for $W \otimes_{D_x} D$.

If $W$ is a simple $D_x$-module, then $W \otimes_{D_x} D$ is a simple $D$-module. Fix $x_\alpha \in \alpha$ for each conjugacy class $\alpha \subset G$. Let $\{W_j\}$ be a complete set of simples for $D_{x_\alpha}$. (The index $j$ depends on $x_\alpha$.) Then

$$V_{x_\alpha,j} := W_j \underset{D_{x_\alpha}}{\otimes} D$$

forms a complete a set of simple $D$-modules.

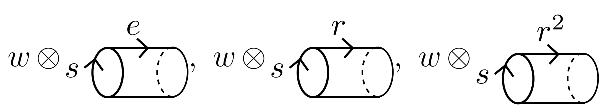
In all of the following, write

$$S_3 \cong \langle s, r | s^2 = r^3 = (rs)^2 = e\rangle.$$

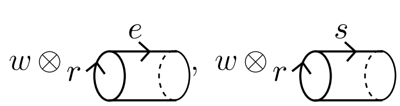
**Example 1.** There are eight simples in $D(S_3)$:

$$V_{e,0}, V_{e,1}, V_{e,2}, V_{s,0}, V_{s,1}, V_{r,0}, V_{r,1}, V_{r,2}.$$

For example, if $w$ is a basis for the nontrivial rep of $D_s$, then




form a basis for $V_{s,1}$. And if $w$ is a basis for the "first" nontrivial rep of $C_r$, then



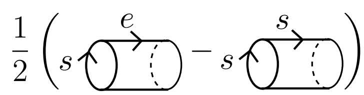

is a basis for $V_{r,1}$. Here the 1 comes from calling the representation $r \mapsto \zeta$ the "first" representation and $r \mapsto \zeta^2$ the "second."

**Example 2.** $V_{r,1}$ is an example of interesting phenomenon. If you restrict $V_{r,1}$ to $D_r$ and use the canonical isomorphism $D_r \cong \mathbb{C}C_r$, you get back the representation $W_1$ of $C_r$. If you restrict to $D_{r^2}$ and use the canonical isomorphism $D_{r^2} \cong \mathbb{C}C_{r^2} = C_r$, you get back the representation $W_2$. This has to do with the fact that $s$ conjugates $r$ to $r^2$ but acts as an outer automorphism on $C_r = C_{r^2}$.

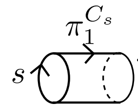
Write

$$\pi_1^{C_s} = \frac{e-s}{2}$$

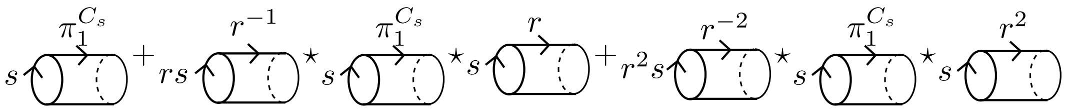
to denote the nontrivial projector in $C_s$. Then




is the nontrivial projector in $D_s$. Write it as

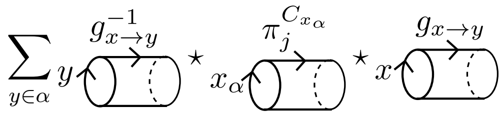



It is then not hard to see that

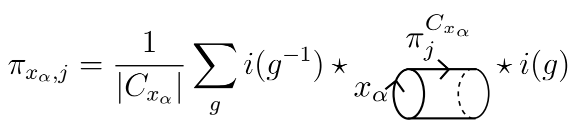



is the projector for $\pi_{s,1} \in D$.

More generally,

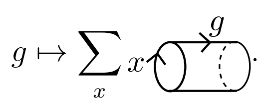



is the projector $\pi_{x_\alpha,j}$. This shouldn't depend on the choices $g_{x_\alpha\to y}$ and in fact it doesn't:




where

$$i : \mathbb{C}G \to D$$




In the case of the group algebra, one can define an inner product on characters by

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V,W) = \dim(V^* \otimes W)^G.$$

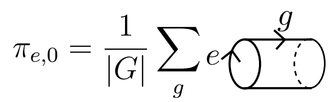
Since the coproduct is $g \mapsto g \otimes g$ and the antipode is $g \mapsto g^{-1}$, $(\chi_V, \chi_W)$ is $\chi_{V^*\otimes W}$ applied to $\pi_0$. In other notation, read left to right,

$$\langle \pi_0 \Delta(S \otimes \mathrm{id}), \chi_V \otimes \chi_W\rangle.$$

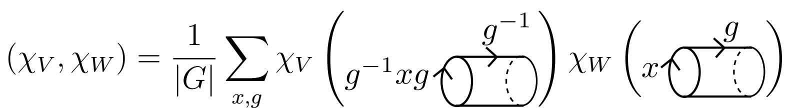
Since Schur's lemma holds for $D$, for $D$-modules $V$ and $W$, define

$$\begin{aligned}(\chi_V, \chi_W) &:= \langle \pi_0, \chi_{V^*\otimes W}\rangle = \dim(V^* \otimes W)^G = \dim(\mathrm{Hom}_G(V,W)) \\ &= \langle \pi_{e,0}\Delta(S\otimes \mathrm{id}), \chi_V \otimes \chi_W\rangle.\end{aligned}$$

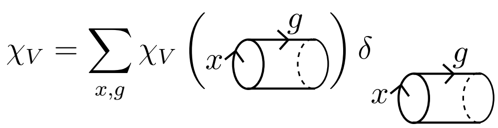
Since



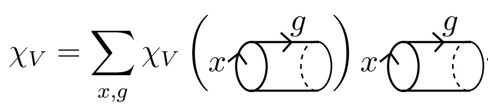
the inner product of characters can be written explicitly:



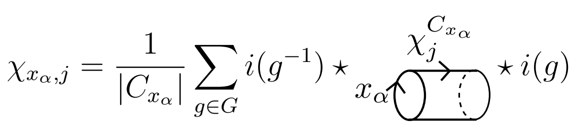
In this form, the inner product can be extended to all functions on $G \times G$. A character can of course be written



Abuse notation, though, and write it as

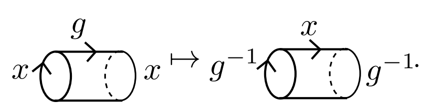


If the cylinder on the right is interpreted with the $-$ orientation, then this is consistent with $\chi_V$ as an element in $D^*$. The following identity follows holds:

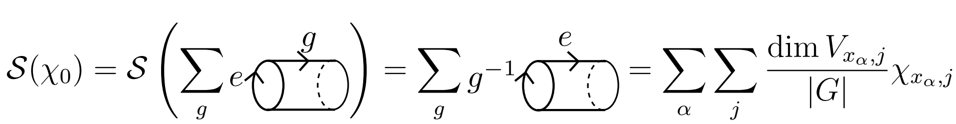


where $\chi_j^{C_{x_\alpha}} = \frac{|C_{x_\alpha}|}{\dim W_j}\pi_j^{C_{x_\alpha}}$.

There's an automorphism of $Z(T^2)$ given by

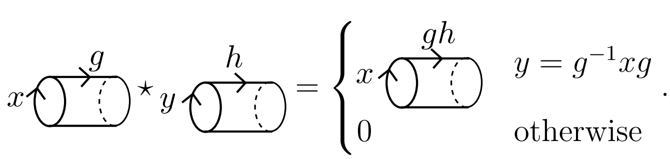


This is (essentially) induced from the "rotate 90 degrees" map on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. It is part of a more general $\mathrm{SL}_2\mathbb{Z}$ action on $Z(T^2)$. Let $\mathcal{S}$ denote the restriction of this map to the characters. For example

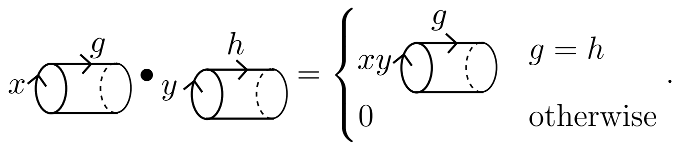


(the last equality can be seen by using the orthogonality of the characters). The map $\mathcal{S}$ provides an interesting automorphism of the characters that has no analog in the usual group algebra $\mathbb{C}G$.

As usual, let $\star$ denote "horizontal multiplication":



Define a new multiplication $\bullet$ called (aptly) "vertical multiplication":




Then $\chi_{V\otimes W} = \chi_V \bullet \chi_W$. In particular, indexing irreps of $D$ by $i$,

$$\chi_i \bullet \chi_j = \sum_k n_{ij}^k \chi_k$$

where $n_{ij}^k$ is the multiplicity of $V_k$ in $V_i \otimes V_j$. And

$$\chi_i \star \chi_j = \begin{cases} \frac{|G|}{\dim V_i}\chi_i & i = j \\ 0 & \text{otherwise} \end{cases}.$$

To make an analogy to the group algebra, $\bullet$ is like the usual multiplication of characters, and $\star$ is like the convolution of characters.

In classical Fourier analysis, the Fourier transform interpolates between multiplication of functions and convolution of functions. $\mathcal{S}$ plays the same role:

$$\mathcal{S}(\chi_i \bullet \chi_j) = \mathcal{S}(\chi_i) \star \mathcal{S}(\chi_j).$$

This identity immediately follows from the interpretation of $\mathcal{S}$ as rotation by 90 degrees.

Exercises:

1. Define a multiplication $*$ on $(\mathbb{C}G)^*$ via

$$\delta_g * \delta_h = \delta_{gh}.$$

   (a) Show that there's an algebra isomorphism between $\mathbb{C}G$ (with the usual multiplication) and $(\mathbb{C}G)^*$ (with $*$). What do the projectors in $\mathbb{C}G$ correspond to?

   (b) Let $\chi_i$ be characters of $\mathbb{C}G$. Use the isomorphism from (a) to show that

$$\chi_i * \chi_j = \begin{cases} \frac{|G|}{\dim V_i}\chi_i & i = j \\ 0 & \text{otherwise} \end{cases}.$$

   (c) Now let $i$ index the simple modules of $D(G)$. Show that

$$\chi_i \star \chi_j = \begin{cases} \frac{|G|}{\dim Vi}\chi_i & i = j \\ 0 & \text{otherwise} \end{cases}.$$

2. Compute the $8 \times 8$ matrix for $\mathcal{S}$ mapping the characters of $D(S_3)$ to itself.

3. Let $i$ index the simple modules of $D(G)$. Write

$$V_i \otimes V_j \cong \bigoplus_k V_k^{n_{ij}^k}.$$

Compute $n_{ij}^k$ in terms of the matrix entries $\mathcal{S}_a^b$ of $\mathcal{S}$: $\mathcal{S}(\chi_a) = \sum_b \mathcal{S}_a^b \chi_b$, and the entries of its inverse.