A unital associative algebra (over $\mathbb{C}$, say) is a vector space A together with a map

$$i : \mathbb{C} \to A$$

and a map

$$m : A \otimes A \to A$$

that satisfy a left unit axiom

$$\begin{array}{ccc} A = \mathbb{C} \otimes A & \xrightarrow{\mathrm{id}_A \otimes i} & A \otimes A \\ & \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle m} \\ & & A \end{array}$$

(and similarly a right unit axiom) and an associativity axiom

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}_A} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array}.$$

Abuse notation and write $1 \in A$ for $i(1)$ and write multiplication ab as $m(a \otimes b)$. Then these axioms translate to

$$1a = a, \ a(bc) = (ab)c.$$

Note that $A \otimes A$ picks up an algebra structure $(a \otimes b)(c \otimes d) = ac \otimes bd$. A Hopf algebra consists of three extra maps

$$\Delta : A \to A \otimes A \text{ algebra morphism}$$

$$\epsilon : A \to \mathbb{C} \text{ algebra morphism}$$

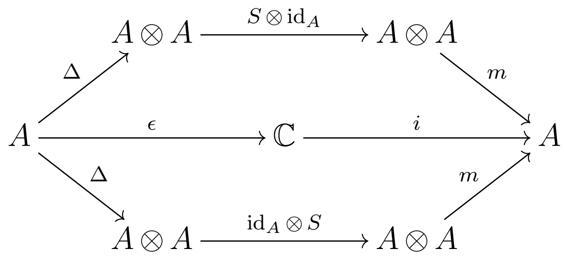
$$S : A \to A \text{ algebra antimorphism}$$

that satisfy the extra commutative diagrams

$$\begin{array}{ccc} A = \mathbb{C} \otimes A & \xleftarrow[\mathrm{id}_A \otimes \epsilon]{} & A \otimes A \\ & \nwarrow_{\mathrm{id}_A} & \uparrow{\scriptstyle \Delta} \\ & & A \end{array}$$

(and also a similar diagram with $A \otimes \mathbb{C}$)

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow[\Delta \otimes \mathrm{id}_A]{} & A \otimes A \\ \uparrow{\scriptstyle \mathrm{id}_A \otimes \Delta} & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow[\Delta]{} & A \end{array}$$



The first two diagrams are obtained from the algebra diagrams, reversing arrows and replacing $i \leftrightarrow \epsilon$ and $m \leftrightarrow \Delta$. The last diagram is self-dual in this sense.

If V and W are right modules of A, then $V \otimes W$ is naturally a right $A \otimes A$-module, but not an A-module. Similarly, V^* is a left A-module, but not a right A-module. The field $\mathbb{C}$ does not necessarily admit an action by A. The point of a Hopf algebra is that $V \otimes W$, V^* and $\mathbb{C}$ become right A-modules. Namely, for $a \in A$:

$$(v \otimes w) \cdot a := (v \otimes w) \cdot \Delta(a),\ v \in V,\ w \in W$$

$$\langle v, \xi \cdot a \rangle := \langle v \cdot S(a), \xi \rangle,\ v \in V, \xi \in V^*$$

$$z \cdot a := \epsilon(a) z,\ z \in \mathbb{C}.$$

The counitality axiom, for example, says that the natural map

$$V \otimes \mathbb{C} \to V$$

$$v \otimes 1 \mapsto v$$

is a map of A-modules and the coassociativity axiom says that

$$V \otimes (W \otimes U) \to (V \otimes W) \otimes U$$

$$v \otimes w \otimes u \mapsto v \otimes w \otimes u$$

is a map of A-modules.

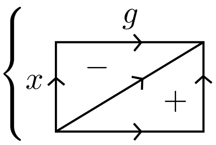
If A is finite dimensional, then A^* is a Hopf algebra with multiplication Δ^*, unit ϵ^*, comultiplication m^*, counit i^*, and antipode S^*.

The 2D theory assigns to I a vector space and assigns to $-\Delta^2$ a multiplication on that vector space. Hence I is assigned an algebra. Similarly, the 3D theory will assign to $I \times S^1$ a vector space and and will assign to $-\Delta^2 \times S^1$ a multiplication on that vector space. Hence $I \times S^1$ will be assigned an algebra. In fact, this algebra will be a Hopf algebra. Each part of the Hopf algebra structure can be seen topologically. This is the content of what follows.

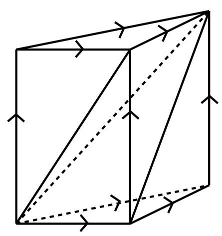
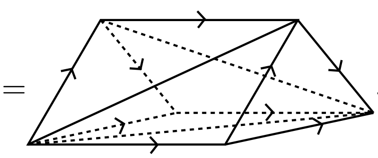
Redefine $D = D(G)$ to be the vector space

$$D := Z(I \times S^1, +)$$

where here the + orientation is as follows[1]:

$$D = \mathbb{C}\left\{ \right.$$  $$\left. \right\}_{x,g\in G}.$$

This orientation is part of a more general statement that $\Delta^n \times \Delta^m$ inherits a Δ-complex structure plus an orientation. The orientation I'm calling "+" on $I \times S^1$ is the canonical one inherited from $I \times I = \Delta^1 \times \Delta^1$. Rather than describe the oriented Δ-complex structure on $\Delta^n \times \Delta^m$, I will work on a case by case basis. Here are the Δ-complex structures on $I \times \Delta^2$ and $\Delta^2 \times I$:

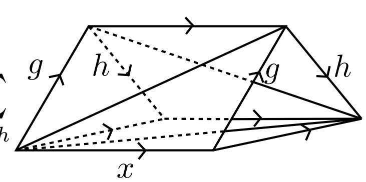
$I \times \Delta^2 =$  $\Delta^2 \times I =$  .

The positive orientation on $I \times \Delta^2$ is the following: read left to right, the three 3-simplices are oriented +, −, +. The positive orientation on $\Delta^2 \times I$ is the following: read fron to back, the three 3-simplices are oriented +, −, +.

The following two points can be checked:

- In $\partial(I \times \Delta^2)$, the two front rectangles get the − orientation and the back rectangle gets the + orientation. The top triangle is positive and the bottom triangle is negative.
- In $\partial(\Delta^2 \times I)$, the two top rectangles get the + orientation and the bottom rectange gets the − orientation. The left and right triangles are oriented − and +, respectively.

$\Delta^2 \times S^1$ inherits a Δ-complex structure from $\Delta^2 \times I$, obtained by gluing together the two triangles on the side. Then

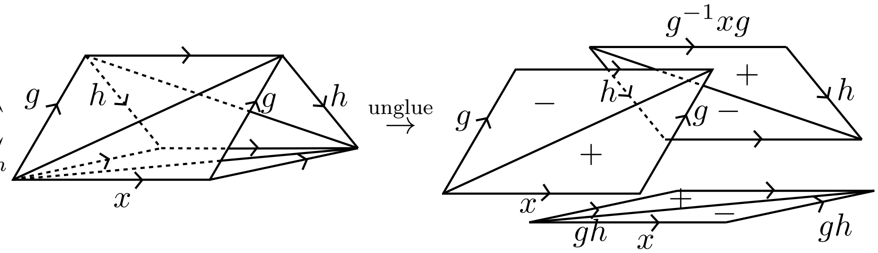
$$Z(-\Delta^2 \times S^1) = \sum_{x,g,h}$$ 

where the picture on the right is to be thought of as the boundary of $\Delta^2 \times S^1$ (it's hard to indicate via illustration that it shouldn't be filled in). After

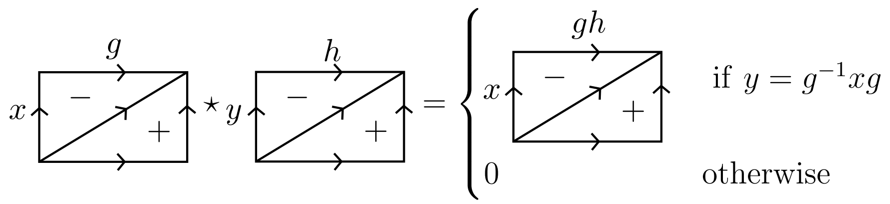
[1] In the picture of the rectangle, the top and bottom edges should be identified (in particular they are both labeled g).

ungluing then,

$$Z(-\Delta^2\times S^1) = \sum_{x,g,h}$$

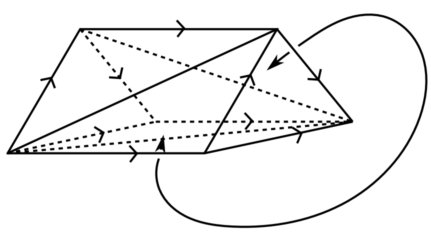


This is in $D^*\otimes D^*\otimes D$, and hence gives a multiplication in D:

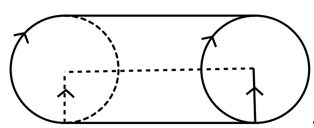


In these pictures, the top and bottom edges of the rectangles should be thought identified.

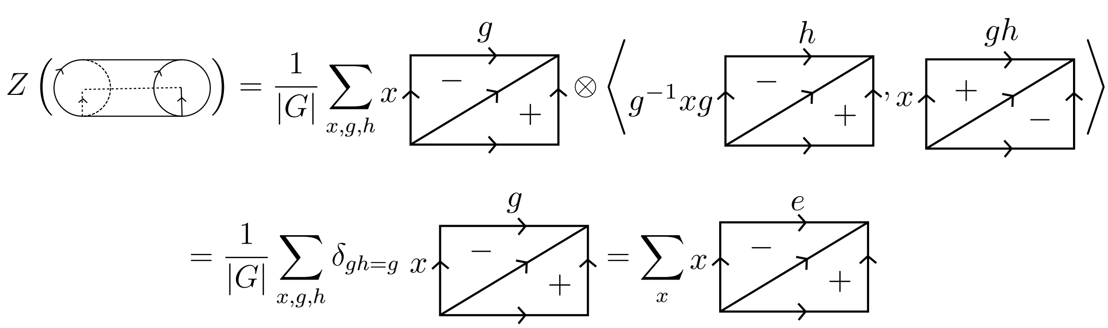
Similarly to the $2D$ theory, the identity element can be obtained by gluing together two sides of $\Delta^2\times S^1$. Namely make the following identification (side triangles still thought to be identified):



to get

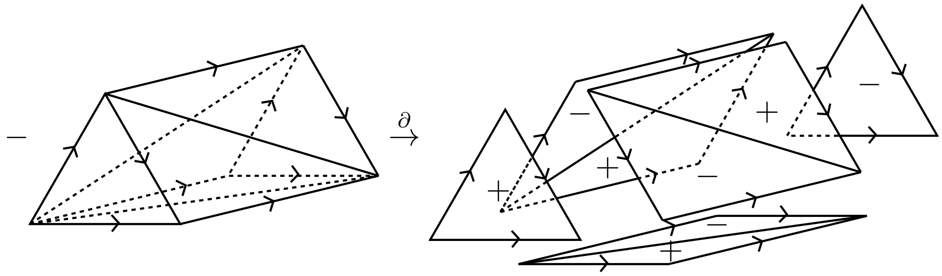


Then by the gluing theorem,



the identity in D. The factor of $\frac{1}{|G|}$ comes from a newly interior vertex.

Let $(I\times\Delta^2)'$ denote $I\times\Delta^2$ with the three edges $(\Delta^2)^0\times I$ identified. Then (note negative orientation)

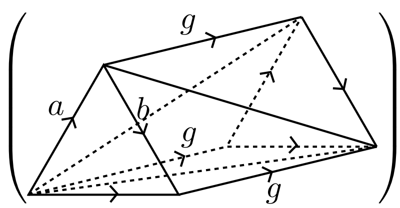


(with the three horizontal edges identified). Thus, after ungluing,

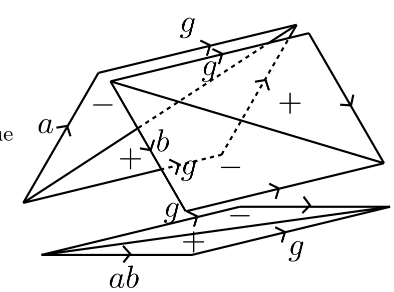
$$Z(-(I\times\Delta^2)')\in D^*\otimes D\otimes D\otimes Z(\Delta^2)\otimes Z(\Delta^2)^*.$$

This is not a coproduct on the nose, since a coproduct would live in $D^*\otimes D\otimes D$, but in this setting one can forget the triangles on the end and no harm will come of it. Thus, consciously abusing notation, write

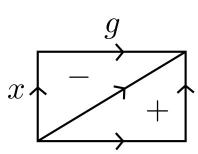
$$Z(-(I\times\Delta^2')) = \sum_{a,b,g}\partial$$



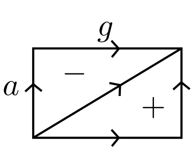
$\overset{\text{unglue}}{\longrightarrow}$



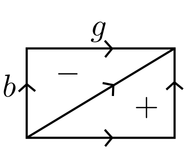
Written as a map $D\to D\otimes D$, $Z(-(I\times\Delta^2)')$ is therefore



$$\mapsto \sum_{ab=x} a$$



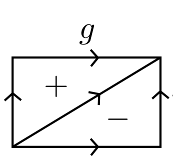
$\otimes\, b$



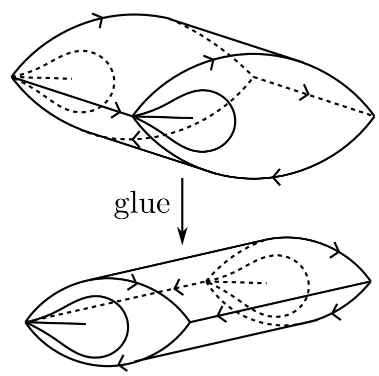
Again, the top and bottom of these rectangles should be thought identified.

The counit is related to the solid torus but with the roles of the meridian and longitude switched, thus

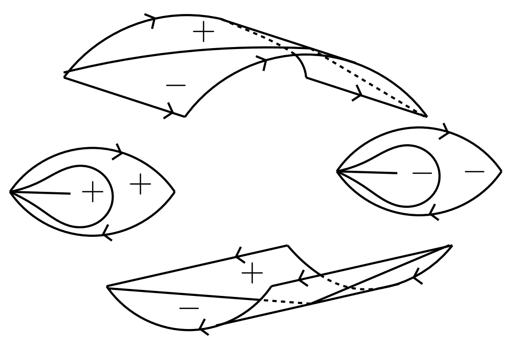
$$\epsilon = \sum_g e$$



The complex for the antipode is constructed from two complexes:

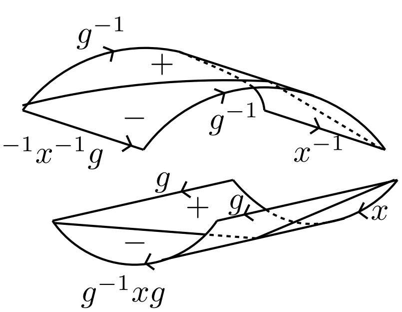


The first is obtained from two copies of $\Delta^2 \times I$ and second is obtained from two copies of $I \times D^2$. Identify the left and right sides together. Call the result K_0 for the moment. The boundary of the resulting complex is (with orientations indicated)



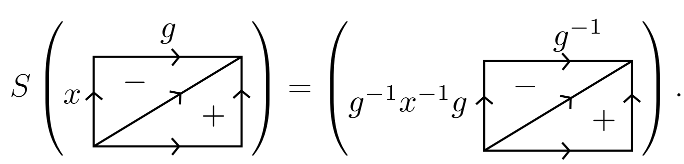
but, as before, forget the bigons on the ends. Then, consciously abusively,

$Z(K_0) = \sum_{x,g} g^{-1}x^{-1}g$



$\in D^* \otimes D.$

As a map $D \to D$, S is



Exercise:

1. (a) Write out the Hopf algebra structure for $\mathbb{C}G$ (e.g., unit, multiplication, counit, antipode, coproduct).

(b) Since $\mathbb{C}G$ is a Hopf algebra, $(\mathbb{C}G)^*$ is also a Hopf algebra. What is it isomorphic to? What are the simple modules of $(\mathbb{C}G)^*$ and how do they behave under tensor product?

(c) (for fun) 2D Dijkgraaf-Witten spits out the algebra $\mathbb{C}G$, but just the algebra structure. Is there any way to work out the Hopf algebra structure on $\mathbb{C}G$ from the topology of 2-complexes?

2. Let A be a Hopf algebra. Show that the counitality axiom implies that $V \cong V \otimes \mathbb{C}$ as A-modules.

3. One of the axioms of a Hopf algebra is that (composition left to right)

$$\Delta \circ S \otimes \mathrm{id} \circ m = \epsilon \circ i = \Delta \circ \mathrm{id} \otimes S \circ m.$$

What property does this imply for the modules of A? (It might help to consider the case of $\mathbb{C}G$ first.)

4. In a Hopf algebra A, Δ must be an algebra map, i.e.,

$$m_A \circ \Delta = \Delta \otimes \Delta \circ m_{A \otimes A}.$$

Show this holds for $D(G)$ by exhibiting a Δ-complex for each side.