Last time covered 2-manifolds and 2D Dijkgraaf-Witten theory. This lecture covers 3-manifolds and 3D Dijkgraaf-Witten theory.

Here are some examples of 3-manifolds:

Example 1. S^3 is the unit sphere in $\mathbb{R}^4$. It is homeomorphic to the one-point compactification of $\mathbb{R}^3$. (Compare with S^2 being the one-point compactification of $\mathbb{R}^2$.)

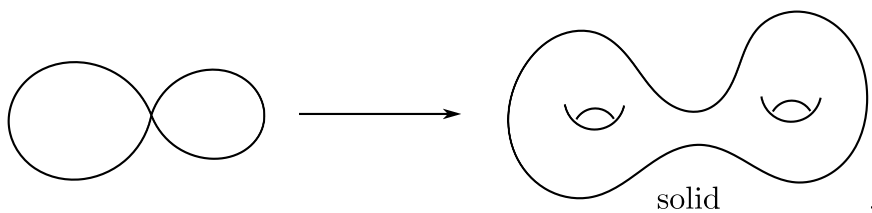
Example 2. An oriented Δ-complex structure on S^3: take two oppositely oriented 3-simplices and glue them together to obtain S^3. Compare with gluing together two oppositely oriented 2-simplices to get the 2-sphere.

Example 3. Write D^n for the set of points of norm at most 1 in $\mathbb{R}^n$ (D stands for Disk, a la D^2). The boundary operation $X \mapsto \partial X$ satisfies a Leibniz rule. In particular,

$$S^3 = \partial(D^4) \cong \partial(D^2 \times D^2) = (\partial D^2) \times D^2 \cup_{\partial D^2 \times \partial D^2} D^2 \times (\partial D^2)$$

so S^3 can be realized as two solid tori glued together. If you a hold a donut in $\mathbb{R}^3$, the space outside the donut, plus a point at infinity, forms another donut.

Example 4. More generally than a donut, take a small neighborhood of a graph in $\mathbb{R}^3$:



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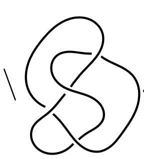
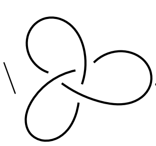
These 3-manifolds are called handlebodies.

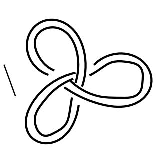
Example 5. Remove a knotted string from S^3:

$S^3 \setminus$  . (1)

or

$S^3 \setminus$  . (2)

Or remove an small neighborhood of the string

$S^3 \setminus$ 

to get a manifold with torus boundary.

Example 6. Let Σ denote a 2-manifold. $S^1 \times \Sigma$ can be described by the quotient

$$(I \times \Sigma)/((0,x) \equiv (1,x)).$$

More generally, for $f : \Sigma \to \Sigma$ a homeomorphism let

$$M_f = (I \times \Sigma)/((0,x) \equiv (1,f(x))).$$

This is a cylinder over Σ with the two ends glued together in a complicated way.

Example 7. In the previous example, let $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. $\mathrm{SL}_2\mathbb{Z}$ acts on $\mathbb{R}^2$ by homeomorphisms and sends $\mathbb{Z}^2$ to itself—so $\mathrm{SL}_2\mathbb{Z}$ acts on T^2 by homeomorphisms. One can let A be any matrix in $\mathrm{SL}_2\mathbb{Z}$ and form the closed 3-manifold M_A.

Example 8. Same as previous example, except let $\Sigma = T^2 \setminus 0$. $\mathrm{SL}_2\mathbb{Z}$ still acts since it preserved 0. It turns out that (2) is homeomorphic to

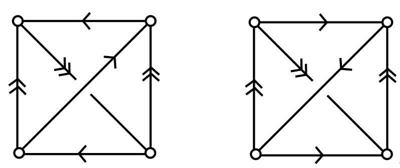
$$M_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)}$$

and (1) is homeomorphic to

$$M_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)}.$$

This is not at all obvious.

Example 9. Here's another even less obvious description of (1). Take two 3-simplices and remove the vertices. (In this situation the vertices are not considered to be ordered.) Glue the two simplices together in the unique way that the following arrows match:

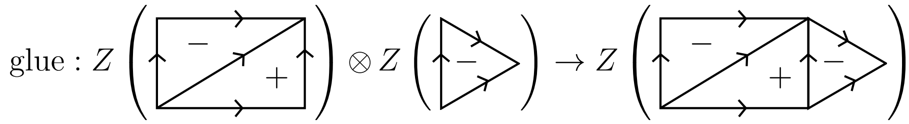


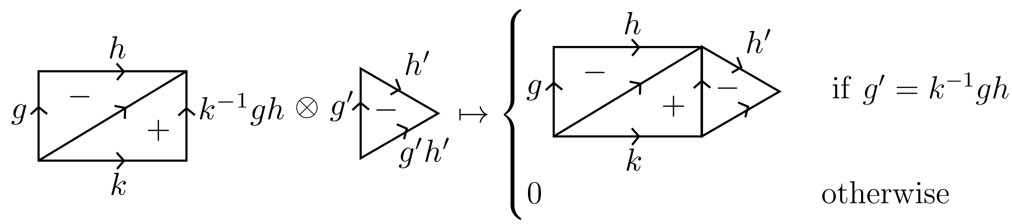
The resulting manifold is homeomorphic to (1).

The construction of 3D Dijkgraaf-Witten theory is completely analogous to the construction of the 2D theory. One significant difference is that the gluing and ungluing maps are not isomorphisms.

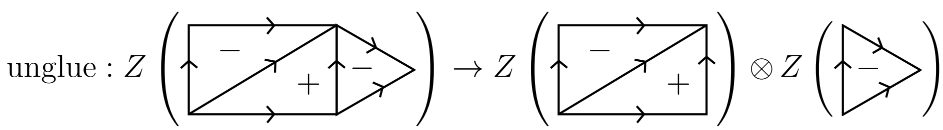
Definition 10. 3D Dijkgraaf-Witten theory is the following data. Given (L, x_L) an oriented 2-manifold with Δ-complex structure, let $Z(L, x_L) = \mathbb{C}\,\mathrm{Hom}(L, G)$. Define the following operations on $Z(L, x_L)$:

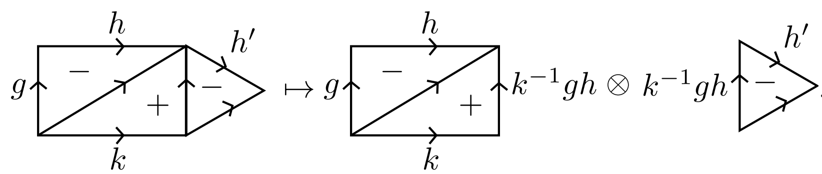
- A gluing map given by sticking together G-labelings along oppositely oriented portions of ∂L, e.g.,





- An ungluing map that splits a G-labeling on a subcomplex of ∂L into two, e.g.,





- A pairing map between oppositely oriented surfaces

$$Z(L, x_L) \otimes Z(L, -x_L) \to \mathbb{C}$$

$$\phi \otimes \psi \mapsto \begin{cases} 1 & \phi = \psi \\ 0 & \text{otherwise} \end{cases}.$$

Given (K, x_K) an oriented 3-manifold with Δ-complex structure, define

$$Z(K, x_K) := \frac{1}{|G|^{\#\text{interior vertices}}} \sum_{\phi \in \operatorname{Hom}(K,G)} \partial\phi \in Z(\partial K, \partial x_K).$$

This ends the definition of 3D Dijkgraaf-Witten theory. It is not quite the same definition that Dijkgraaf and Witten originally gave. See the end of this lecture.

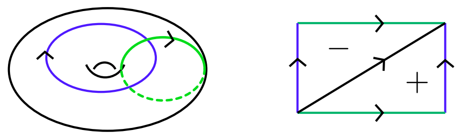
Remark 11. As with the 2D theory, for K closed, then

$$Z(K, x_K) = \frac{|\operatorname{Hom}(\pi_1|K|, G)|}{|G|}$$

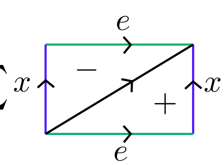
and so does not depend on the orientation x_K.

Remark 12. The two big theorems about the 2D theory port over to the 3D theory: 1) $Z(K, x_K)$ does not depend on the interior Δ-complex structure, and 2) the Gluing Theorem.

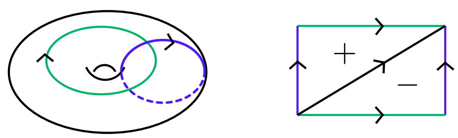
Example 13. Let (K, x_K) be the solid torus with the following oriented Δ-complex structure on the boundary:



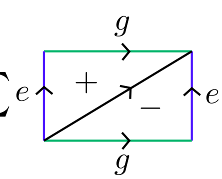
Then

$$Z(K, x_K) = \sum_x$$ 

Example 14. Let (K, x_K) be the solid torus with the following oriented Δ-complex structure on the boundary:

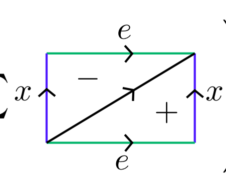
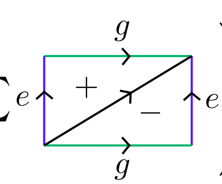


Then

$$Z(K, x_K) = \sum_g$$ 

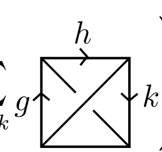
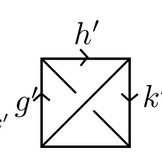
Example 15. Of course $Z(S^3) = \frac{1}{|G|}$ since $\pi_1 S^3$ is the trivial group. Here's another to see it.

Take the two solid tori from the previous two examples. Glue them together respecting the colors. This produces S^3 with one interior vertex. Therefore by the gluing theorem

$$Z(S^3) = \frac{1}{|G|} \left\langle \left(\sum_x \right.\right.$$  $$\left.\right), \left(\sum_g \right.$$  $$\left.\left.\right)\right\rangle = \frac{1}{|G|}$$

(the only term that contributes is the one with $x = g = e$).

Example 16. Think of S^3 as two oppositely oriented 3-simplices glued together. There will be four interior vertices:

$$Z(S^3) = \frac{1}{|G|^4} \left\langle \left(\sum_{g,h,k} \right.\right.$$  $$\left.\right), \left(\sum_{g',h',k'} \right.$$  $$\left.\left.\right)\right\rangle = \frac{|G|^3}{|G|^4} = \frac{1}{|G|}$$

(the only terms that contribute are those with $g = g'$, $h = h'$, $k = k'$).

For a complex K recall that K^0 denotes the 0-simplices of K. Then there's a G^{K^0} action on $\operatorname{Hom}(K,G)$ and hence $\mathbb{C}\operatorname{Hom}(K,G)$.

Let (L, x_L) be an oriented surface. Let $x_I x_L$ denote the orientation on $I \times L$. In particular $\partial(x_I x_L)$ is a positive orientation on $\{1\} \times L$ and a negative orientation on $\{0\} \times L$. Inspection of G-labelings shows that

$$Z(I \times L, x_I x_L) = |G|^{|L^0|} \times \text{projection to trivial part of } G^{L^0} \text{ action.}$$

Let (K, x_K) be an oriented 3-manifold, then

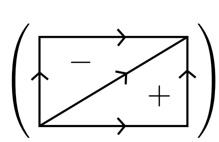
$$Z(K, x_K) = \frac{1}{|G|^{\#\text{interior vertices}}} \sum_{\phi \in \operatorname{Hom}(K,G)} \partial\phi$$

is in the trivial part of $G^{(\partial K)^0}$ action, because acting by G^{L^0} permutes the terms in the sum.

Remark 17. Because the invariant for the 3-manifold always sits in the G^{L^0}-trivial part of $Z(L, x_L)$, for a closed surface Σ, Dijkgraaf and Witten define $Z(\Sigma)$ to be the G-trivial part of $\mathbb{C}\operatorname{Hom}(\pi_1\Sigma, G)$. For M a 3-manifold, $Z(M)$ is defined as above. They also choose put a factor of $\frac{1}{|G|}$ in the pairing map. With these conventions, $Z(I \times \Sigma)$ is assigned the identity map $Z(\Sigma) \to Z(\Sigma)$ and so their theory gives a functor from the category of 3-cobordisms to the category of vector spaces.

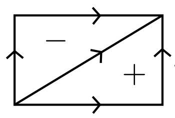
Exercises:

1. Compute $Z(S^1 \times S^2)$.
2. Let X be path connected. Show that $\operatorname{Hom}(\pi_1(X; S), G)/(G^S)$ is in bijection with $\operatorname{Hom}(\pi_1(X, x_0), G)/G$.
3. (a) Let (K, x_K) be a 3-manifold with boundary $(L, x_L) \sqcup (L, -x_L)$. Therefore
   $$Z(K, x_K) : Z(L, x_L) \to Z(L, x_L).$$
   Let K' be the result of gluing the two parts of ∂K together. Show that
   $$Z(K', x_K) = \frac{1}{|G|^{|L^0|}} \operatorname{tr}(Z(K, x_L)).$$
   (b) Show that $Z(S^1 \times L, x_{S^1} x_L) = |\operatorname{Hom}(\pi_1|L|, G)/G|$.
   (c) Compute $Z(S^1 \times S^2)$ again.
4. Observe that

   $D(G) \cong Z$ 

   where the top and bottom edges are identified.

Identify opposite sides of the following to put an oriented Δ-complex structure on T^2:



Call this Δ-complex structure (T^2, x_{T^2}).

(a) How does the composition

$$D(G) \stackrel{\text{glue}}{\to} Z(T^2, x_{T^2}) \stackrel{Z(I\times T^2, x_I x_{T^2})}{\to} Z(T^2, x_{T^2}) \stackrel{\text{unglue}}{\to} D(G)$$

behave when restricted to the center of $D(G)$?

(b) Unglue $Z(T^2, x_{T^2})$ in two different ways to provide maps

$$Z(T^2, x_{T^2}) \to D(G).$$

$$Z(T^2, x_{T^2}) \to D(G)^*.$$

Show that the first restricts to an isomorphism

$$Z(T^2, x_{T^2})^G \to \text{center}(D(G)).$$