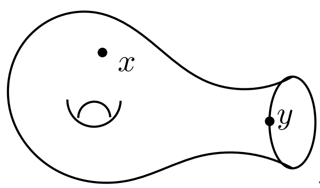
**Definition 1.** A manifold with boundary is a (Hausdorff, second countable) topological space $X$ such that $\forall x \in X$ there exists an open $U \ni x$ and a homeomorphism $\phi_U : U \to \mathbb{R}^n$ or a homeomorphism $\phi_U : U \to \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

Informally, a manifold is a space where every point has a neighborhood homeomorphic to Euclidean space. Think about the surface of the earth. Locally when we look around it looks like $\mathbb{R}^2$, but globally it is not. The surface of the earth is, of course, homeomorphic to the space $S^2$ of unit vectors in $\mathbb{R}^3$. If a point has a neighborhood homeomorphic to $\mathbb{R}^n$ then it turns out intuition is correct and $n$ is constant on connected components of $X$. Typically $n$ is constant on all of $X$ and is called the dimension of $X$. A manifold of dimension $n$ is called an "$n$-manifold."

The boundary of the manifold $X$, denoted $\partial X$, is the set of points which only admit neighborhoods homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

For example, here's a (2-dimensional) manifold with boundary:



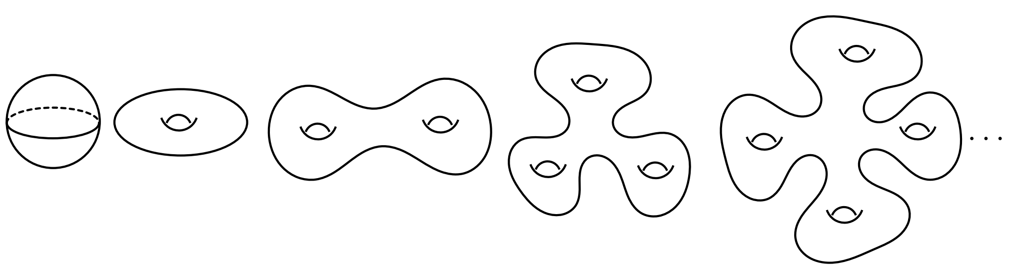

(1)

The point $x$ is in the interior (it has a neighborhood homeomorphic to $\mathbb{R}^2$) and the point $y$ is on the boundary (it has a neighborhood homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}$.)

The term "manifold with boundary" is standard but somewhat confusing. It is possible for a "manifold with boundary" to have empty boundary, i.e., no boundary. I will simply say "manifold" to mean "manifold with boundary." All manifolds are going to be compact.

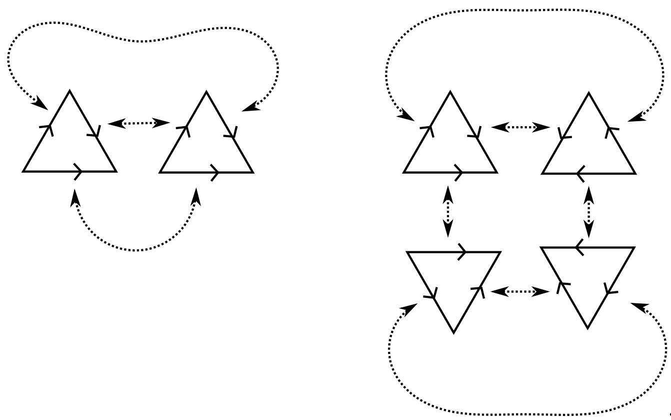
A manifold is called "closed" if it is compact and without boundary. An example of a closed 2-manifold is $S^2$. An example a non-closed 2-manifold is $\mathbb{R}^2$.

**Example 2.** The closed 2-manifolds all look like one of the following:

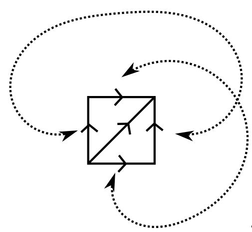


The one with $g$ "holes" is called the closed surface with genus $g$, denoted $\Sigma_g$. For example $\Sigma_0$ is $S^2$ and $\Sigma_1$ is $T^2$.

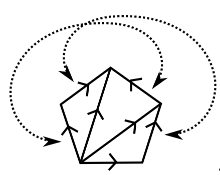
Recall that a $\Delta$-complex can be thought of as a space obtained by gluing together simplices. Here are two different ways of gluing together 2-simplices to get a manifold homeomorphic to $S^2$:



Here's a $\Delta$-complex structure on the torus:



And here's a $\Delta$-complex structure on the torus minus a disk, (1),



(2)

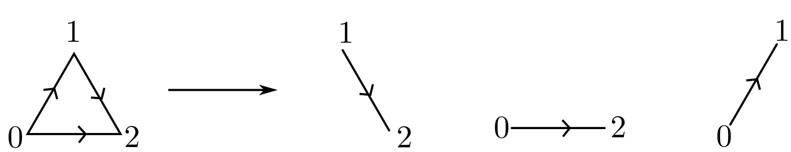
Gluing two of these together gives a $\Delta$-complex on the closed surface of genus 2.

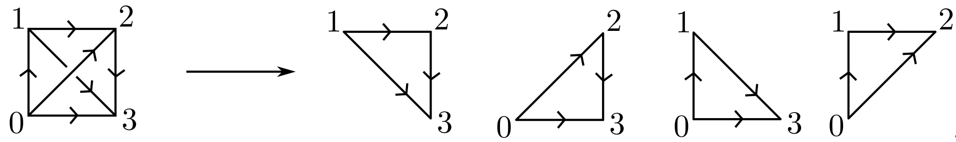
Given a $\Delta$-complex $K$, recall that $|K|$ denotes the underlying topological space. Note that a $\Delta$-complex structure on a manifold $M$ induces a $\Delta$-complex structure on $\partial M$.

**Definition 3.** Let $K$ be a $\Delta$-complex. Let $C_n(K)$ be the free abelian group on the set of $n$-simplices of $K$. $C_n(\partial K)$ sits naturally as a subgroup of $C_n(K)$. Let $C_n(K, \partial K) := C_n(K)/C_n(\partial K)$. $C_n(K)$ is called the group of $n$-chains in $K$.

If $f : K \to K'$ is a combinatorial equivalence, then $f$ induces a map $C_n(K) \to C_n(K')$. This map will (abusively) also be denoted by $f$.

Observe that there are $n+1$ copies of $\Delta^{n-1}$ in the boundary of $\Delta^n$. Each corresponds to removing one vertex from $\Delta^n$:





Given $\sigma$ an $n$-simplex let $\sigma|_{[01\cdots\widehat{i}\cdots n]}$ denote the boundary $(n-1)$-simplex that forgets the $i$th vertex. Turn the above decompositions into a map on chains:

**Definition 4.** Define $\partial : C_n(L) \to C_{n-1}(K)$ by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[01\cdots\widehat{i}\cdots n]}$$

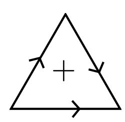
for $\sigma$ an $n$-simplex in $K$.

The reason for the signs is to ensure $\partial \circ \partial = 0$. This is consistent with the idea that "the boundary of a boundary is empty."

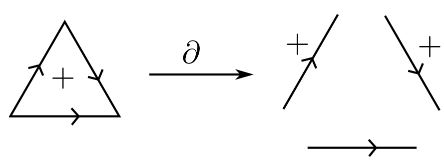
**Example 5.** If $\sigma$ is the unique $n$-simplex in the $\Delta$-complex $\Delta^n$, then

$$\partial\sigma = \sigma|_{[1\cdots n]} - \sigma|_{02\cdots n]} + \cdots + (-1)^n \sigma|_{[01\cdots(n-1)]}.$$

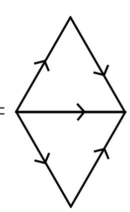
Let



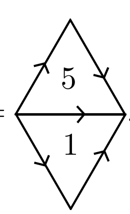
denote $\sigma$ for $n=2$. (Alternatively, put a 1 instead of a + in the center of the triangle.) A negative sign would indicate $-\sigma$. Then for simplicity write the boundary map $\partial : C_2(\Delta^2) \to C_1(\Delta^2)$ as



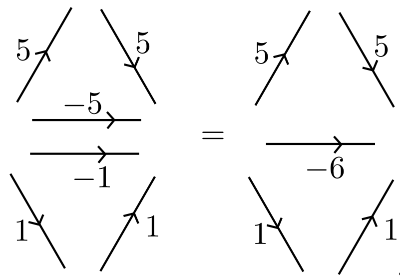
**Example 6.** Let

$K =$ 

and let $x \in C_2(K)$ be the following chain

$x =$ 

Then $\partial x$ is

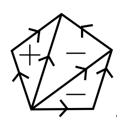



Note that $\partial$ maps $C_n(K) \to C_{n-1}(K)$ but you can project onto $C_{n-1}(K, \partial K)$. The following theorem will not be proved here but is standard in algebraic topology:

**Theorem 7.** *If* $|K|$ *is a connected* $n$*-manifold, then* $\ker(\partial : C_n(K) \to C_{n-1}(K, \partial K))$ *is isomorphic to either* $\mathbb{Z}$ *or* $0$*.*

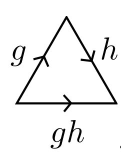
When this kernel is $\mathbb{Z}$, the manifold is called orientable and a generator is called an orientation class. In the case of 0, the manifold called nonorientable. If there are multiple connected orientable connected components, an orientation class is a sum of orientation classes for each component. If $K$ is a complex and $x_K$ an orientation class for $K$, denote the "oriented manifold" by the pair $(K, x_K)$.

Here's a typical example of an orientation class on the one-holed torus (see (2))




The interior edges all cancel in pairs under the boundary map. The image of the boundary map is $-1$ times the boundary 1-simplex. This example illustrates the following observation: if $(K, x_K)$ is an oriented manifold, then $(\partial K, \partial x_K)$ is an oriented manifold.

Recall $\mathrm{Hom}(K, G) = \mathrm{Hom}(\pi_1(K; K^0), G)$ and this set is in bijection with labelings of the edges of $K$ by elements of $G$ such that 2-simplices are labeled like

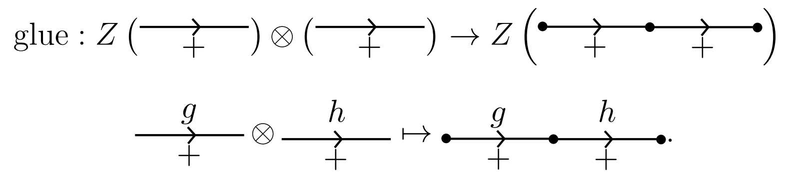



**Definition 8.** 2D Dijkgraaf-Witten theory is the following collection of data.

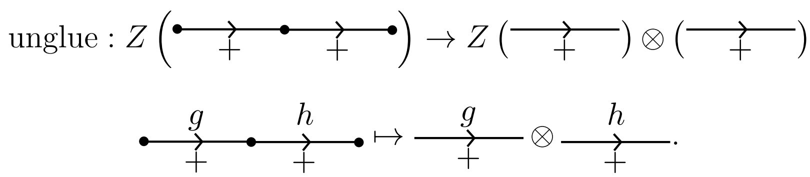
Given an oriented 1-manifold $(L, x_L)$ with $\Delta$-complex structure, let $Z(L, x_L) = \mathbb{C}\,\mathrm{Hom}(L, G)$. It will be helpful to include the empty 1-manifold and set $\mathrm{Hom}(\emptyset, G)$ to be a single element set. $Z(\emptyset, x_\emptyset)$ is canonically isomorphic to $\mathbb{C}$.

Define the following operations on $Z(L, x_L)$:

- A gluing map given by sticking together $G$-labelings along oppositely oriented portions of $\partial L$, e.g.,




- An ungluing map given by splitting together $G$-labelings along a subcomplex




- A pairing map between the vector spaces for oppositely oriented complexes:

$$Z(K, x_L) \otimes Z(L, -x_L) \to \mathbb{C}$$

$$\phi \otimes \psi \mapsto \begin{cases} 1 & \phi = \psi \\ 0 & \text{otherwise} \end{cases}.$$

Let $(K, x_K)$ be an oriented 2-manifold with $\Delta$-complex structure. For $\phi \in \mathrm{Hom}(K, G)$, write $\partial\phi$ for the restriction of $\phi$ to $\partial K$. Use the same letter $Z$ to assign too $(K, x_K)$ an element in $Z(\partial K, \partial x_K)$:

$$Z(K, x_K) := \frac{1}{|G|^{K^0 \setminus (\partial K)^0}} \sum_{\phi \in \mathrm{Hom}(K,G)} \partial\phi \in Z(\partial K, \partial, x_K).$$

This ends the definition of 2D Dijkgraaf-Witten theory.

Note that the gluing and ungluing maps are isomorphisms. This will not be the case in the 3D version.

**Example 9.** Let $(K, x_K)$ be a closed connected 2-manifold. Then

$$Z(K, x_K) = \frac{|\mathrm{Hom}(\pi_1|K|, G)|}{|G|}.$$

In particular, $Z(K, x_K)$ does not depend on the orientation nor the $\Delta$-complex structure for $|K|$.

The previous example is part of the more general theorem:

**Theorem 10.** *$Z(K, x_K)$ only depends on the oriented $\Delta$-complex structure on $\partial K$.*

*Proof.* By construction, $\mathrm{Hom}(K, G) = \mathrm{Hom}(\pi_1(|K|, K^0), G)$ depends only on the vertex set $K^0$ and not on the higher skeletons of $K$. Adding a new point in the interior adds $G$-many new elements to $\mathrm{Hom}(K, G)$. $\square$

$Z(K, x_K)$ does depend on the oriented $\Delta$-complex structure on $\partial|K|$ because, while the underlying vector space for $Z(\partial K, \partial x_K)$ only depends on $(\partial K)^0$ and not on $\partial x_K$, $Z(\partial K, \partial x_K)$ fits into gluing, ungluing, and pairing operations that depend on the oriented $\Delta$-complex structure of $\partial K$.

**Example 11.** Write

$$A := Z\left( \xrightarrow[+]{g} \right).$$

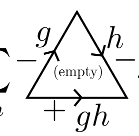
Since the pairing map

$$Z\left( \xrightarrow[+]{g} \right) \otimes Z\left( \xrightarrow[-]{g} \right) \to \mathbb{C}$$

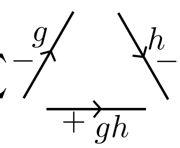
is nondegenerate, it can be used to identify

$$A^* \cong Z\left( \xrightarrow[-]{g} \right).$$

Note that

$$Z(-\Delta^2) = \sum_{g,h}$$  .

After applying an ungluing map, this becomes

$$Z(-\Delta^2) \overset{\text{(unglued)}}{=} \sum_{g,h}$$  $$\in A^* \otimes A^* \otimes A.$$

This gives a multiplication in $A$. In fact, $A$ is canonically isomorphic to the group algebra:

$$A \to \mathbb{C}G.$$
$$\xrightarrow[+]{g} \mapsto g.$$

**Theorem 12** (Gluing Theorem)**.** *Let $(K, x_K)$ be an oriented 2-manifold and suppose $\partial K = L_1 \cup L_2 \cup L_3$ where each 1-simplex in $\partial K$ lies in exactly one of the $L_i$. Write $\partial x_K = x_{L_1} + x_{L_2} + x_{L_3}$ for $x_{L_i} \in C_1(L_i)$. Let $u$ denote the ungluing map*

$$u : Z(\partial K, \partial x_K) \to Z(L_1, x_{L_1}) \otimes Z(L_2, x_{L_2}) \otimes Z(L_3, x_{L_3}).$$

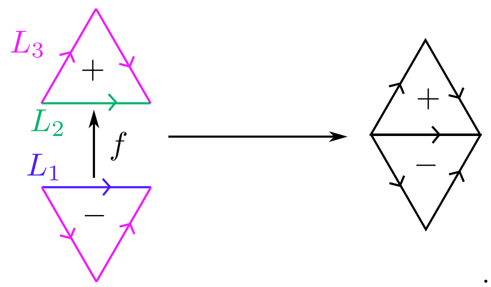
*Suppose that there's a combinatorial equivalence $f : L_1 \to L_2$ such that $f(x_{L_1}) = x_{L_2}$. Let $K'$ be the result of starting with $K$ and gluing $L_1$ to $L_2$ using $f$. Then $x_K$ descends to an orientation class $x_{K'}$ for $K$ and*

$$Z(K', x_{K'}) = \frac{1}{|G|^{\#(\text{newly interior vertices})}} \operatorname{tr}_{12}(u(Z(K, x_K)))$$

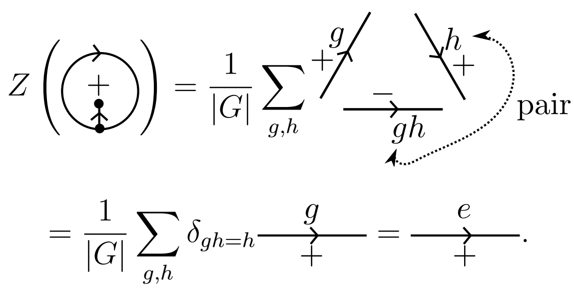
*where* $\mathrm{tr}_{12}$ *denotes the pairing of the* $Z(L_1, x_{L_1})$ *and* $Z(L_2, x_{L_2})$ *factors (using* $f$ *to identify* $Z(L_1, x_{L_1})$ *with* $Z(L_2, -x_{L_2})$*). The phrase "newly interior vertices" means interior vertices in* $(K')^0$ *that do not come from interior vertices of* $K$*.*

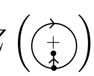
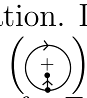
*Proof.* Exercise. □

The picture to have in mind is

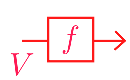



**Example 13.** Start with $\Delta^2$ and glue two sides together. Then



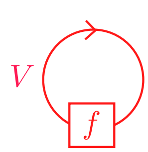

**Remark 14.** $Z(-\Delta^2)$ gives $A$ a multiplication and $Z$  gives a unit for the multiplication. Dually, $Z(\Delta^2)$ gives $A$ a "Frobenius comultiplication" $A \to A \otimes A$, and $Z$  gives $A$ a "Frobenius counit" $A \to \mathbb{C}$. These data give $A$ the structure of a Frobenius algebra. This is different than a bialgebra.

Write




to denote a linear map $f : V \to V$. Alternatively, an element of $V^* \otimes V$. Reading an arrow in reverse is the adjoint map, gluing arrows end to end is composition (equivalently, contraction of $V \otimes V^* \to \mathbb{C}$), and therefore

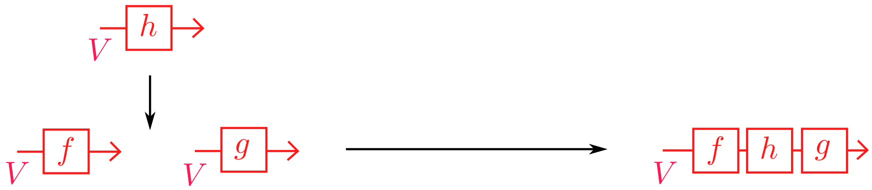

is the identity map $V \to V$ and



is the trace of the map $f : V \to V$. Disjoint union stands for tensor product, so

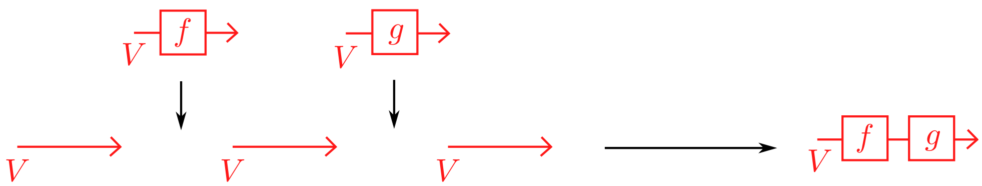
$$(3)$$

stands for $f \otimes g : V \otimes V \to V \otimes V$. One can also, however, think of (3) as a map $\mathrm{End}(V) \to \mathrm{End}(V)$ as follows:



namely, precomposition by $f$ and postcomposition by $g$. Similarly,

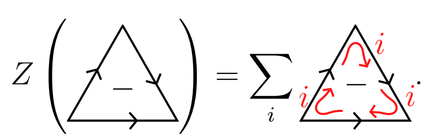
gives composition of endomorphisms



i.e., multiplication in the algebra $\mathrm{End}(V)$. Let $\{V_i\}$ be a collection of vector spaces. Then the multiplication in the algebra $\bigoplus_i \mathrm{End}(V_i)$ is

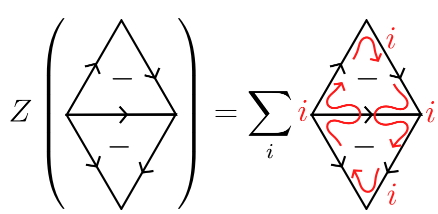
If $\{V_i\}$ is a complete collection of simple modules for $A$, this describes the multiplication for $A$ under the isomorphism $A \cong \bigoplus_i \mathrm{End}(V_i)$.

The red arrows permit the following very elegant notation:

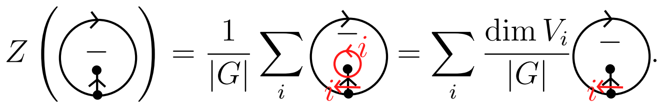


The simplex is drawn on the right side to indicate context (i.e., where are the two inputs and where is the output).

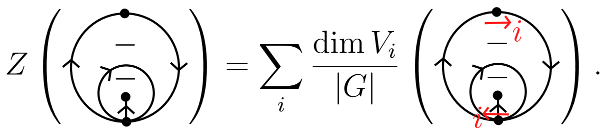
According to the gluing theorem, gluing together simplices along oppositely oriented faces corresponds to contracting $A$ with $A^*$. Under the isomorphism $A \cong \bigoplus_i \mathrm{End}(V_i)$, this translates to gluing the ends of arrows together—if the labeling $i$ matches on the two arrows. For example



and

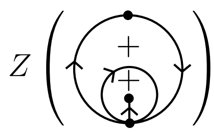


so therefore

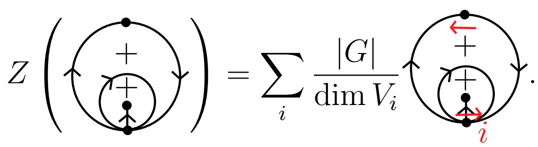


(4)

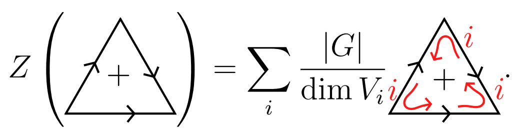
Think of (4) as a map $A \to A^*$. Then its inverse is given by



so that



Gluing two of these onto $-\Delta^2$ and also gluing on a copy of (4), then



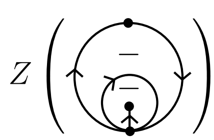
Exercises:

1. Write down a $\Delta$-complex structure for a closed surface of genus $g$. Is it orientable?

2. Write down a $\Delta$-complex structure for the Mobius band. Glue two of these together. Is the result orientable?

3. Let $C$ be the cylinder oriented so the two ends are oriented oppositely. As a map $Z(S^1, +) \to Z(S^1, +)$, what is it?

4. Show that, after ungluing and identifying $Z(I, +)$ with $\mathbb{C}G$, that

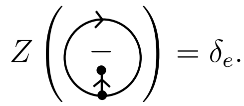
   

   gives a map

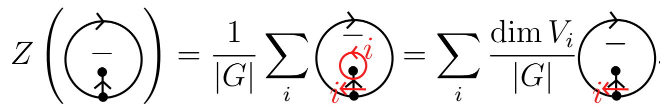
   $$g \mapsto \delta_{g^{-1}}.$$

   Find an oriented complex $K$ that gives the inverse to this map.

5. After identifying $Z(I, -)$ with $(\mathbb{C}G)^*$, show that



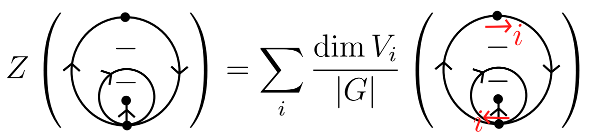
The red arrows indicate that



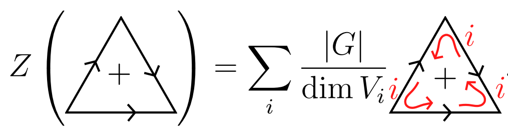
Check using orthogonality of characters the identity

$$\delta_e = \sum_i \frac{\dim V_i}{|G|} \chi_i.$$

6. Show that



7. Prove the assertion (mentioned in the lecture) that



8. Prove Mednykh's formula:

$$\frac{|\operatorname{Hom}(\pi_1 \Sigma_g, G)|}{|G|} = \sum_i \left( \frac{|G|}{\dim V_i} \right)^{2g-2}.$$