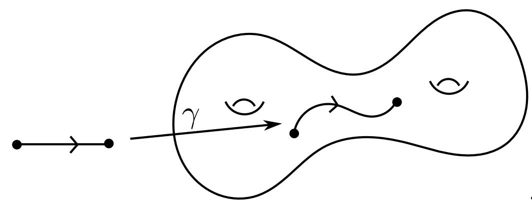
General references for topology are Munkres's book on Topology and Hatcher's book on Algebraic Topology.

Let X be a topological space. Write $I = [0,1]$ for the closed interval. All maps are continuous.

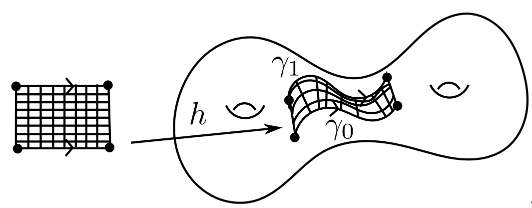
Definition 1. A path in X is a map $\gamma : I \to X$:



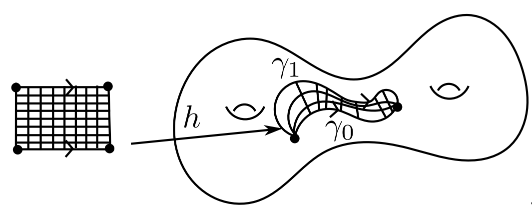
Write $x = \gamma(0)$ and $y = \gamma(1)$. Then γ is often said to be a path from x to y, denoted $x \to y$.

Definition 2. A homotopy from γ_0 to γ_1 is a map $h : I \times I \to X$ such that

$$h(t,0) = \gamma_0(t),\ h(t,1) = \gamma(t)$$



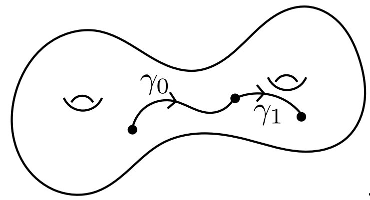
Definition 3. Let γ_0 and γ_1 be two paths from x to y. A homotopy from γ_0 to γ_1 rel endpoints is a homotopy such that $h(0,s) = x$ and $h(1,s) = y$ for all $s \in I$:



Write $\gamma_0 \simeq \gamma_1$ if there is a homotopy from γ_0 to γ_1. In what follows homotopies will always be rel endpoints. Homotopy rel endpoints is an equivalence relation.

Definition 4. Given γ_0 from x_0 to x_1 and γ_1 from x_1 to x_2. Define the concatenation $\gamma_0 * \gamma_1 : I \to X$ by "first move along γ_0 from $t = 0$ to $t = \frac{1}{2}$ then move along γ_1 from $t = \frac{1}{2}$ to $t = 1$" or, formally,

$$(\gamma_0 * \gamma_1)(t) = \begin{cases} \gamma_0(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$



Definition 5. Let $\overline{\gamma}(t) = \gamma(1-t)$. This runs the path γ in reverse.

Definition 6. Let c_x denote the constant path at $x \in X$.

Claim 7. *If γ is a path from x to y, then $\gamma * \overline{\gamma} \simeq c_x$.*

Proof. This proof is best thought of as "run along γ until $\gamma(s)$ and then run backwards to $\gamma(0)$—then take s from 1 to 0." But here it is precisely:

$$\gamma_s(t) := \gamma((1-s)t)$$

$$h(t,s) := (\gamma_s * \overline{\gamma_s})(t)$$

gives the relevant homotopy. □

Claim 8. *Let γ be a path from x to x. Then $\gamma * c_x \simeq \gamma$.*

Proof. This proof is best thought of as "spend a fraction s of the time on γ and the rest at x—then take s from 1 to 0." But here it is precisely:

$$h(t,s) := \begin{cases} \gamma((2-s)t) & 0 \leq t \leq \frac{1}{2-s} \\ x & \frac{1}{2-s} \leq t \leq 1 \end{cases}.$$

□

Let $\pi_1(X,x)$ be the set of homotopy classes of maps $x \to x$.

Proposition 9. *$\pi_1(X,x)$ is a group with multiplication and unit:*

$$[\gamma][\eta] = [\gamma * \eta]$$

$$1_{\pi_1(X,x)} = [c_x].$$

Proof. You can check that multiplication is well-defined: it does not depend on the choice of curve in the homotopy class $[\gamma]$. The previous two claims show that $[\overline{\gamma}]$ is an inverse to $[\gamma]$ and that $[c_x]$ is the identity. It is an exercise to check that

$$((\gamma * \eta) * \eta) \simeq (\gamma * (\eta * \xi))$$

proving associativity. □

$\pi_1(X,x)$ is called "the fundamental group of X."

Definition 10. For $f : (X,x) \to (Y,y)$, let $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ be defined by $f_*([\gamma]) = [f \circ \gamma]$.

It is not hard to see that if x and y are in the same path component of X, then $\pi_1(X, x) \cong \pi_1(X, y)$. Sometimes, if the basepoint is not important, $\pi_1(X, x)$ is (abusively) denoted $\pi_1(X)$.

Example 11. $\pi_1(S^1, x) \cong \mathbb{Z}$. Any loop from x to x is homotopic to the standard curve winding around n times: $t \mapsto e^{2\pi i n t}$ for some $n \in \mathbb{Z}$.

Idea of Proof. Think of S^1 as $\mathbb{R}/\mathbb{Z}$ let x be the class of the integers. Any curve looping around S^1 can be factored as a map

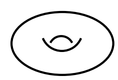
$$I \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1.$$

Because $\mathbb{R}$ is affine, it is easy to straighten out the path $I \to \mathbb{R}$ to be straight between its endpoints. □

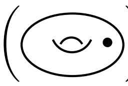
The following is easy to see:

Claim 12. $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$.

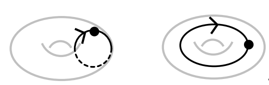
Example 13.

$$S^1 \times S^1 \cong$$ 

so that

$$\pi_1$$ () $$\cong \mathbb{Z} \times \mathbb{Z}.$$

Here are explicit generators:

 .

Given pointed spaces (X, x) and (Y, y) let

$$X \vee Y = (X \sqcup Y)/x \sim y$$

that is, X and Y glued together at $x = y$.

Example 14. I will not prove it here, but it is believable that

$$\pi_1(X \vee Y, x) \cong \pi_1(X, x) * \pi_1(Y, y)$$

the free product of the fundamental groups of X and Y. For example, the fundamental group of the wedge of two circles is the free group on two generators. The two generators are loops going once around each circle.

More generally:

Theorem 15 (Seifert-van Kampen)**.** *Write $X = U \cup V$ for U, V path-connected open sets such that $U \cap V$ is path connected. Denote the inclusions by*

$$i_U : U \cap V \hookrightarrow U$$

$$i_V : U \cap V \hookrightarrow V.$$

Fix $x \in U \cap V$. Then

$$\pi_1(X, x) \cong (\pi_1(U, x) * \pi_1(V, x))/N$$

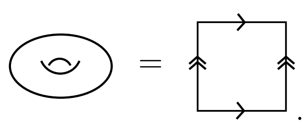
where N is the smallest normal subgroup containing

$$(i_U)_*[\gamma]((i_V)_*[\gamma])^{-1}$$

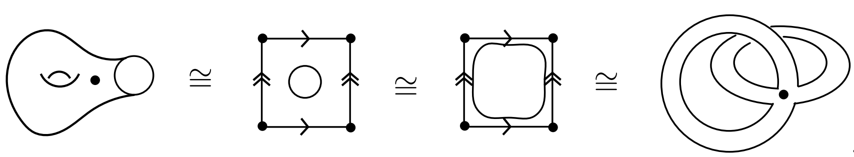
for each $[\gamma] \in \pi_1(U \cap V, x)$.

Informally: the fundamental group of X and Y glued along Z is the free product of the fundamental groups of X and Y, modulo identifying corresponding curves in the copies of Z in X and Y.

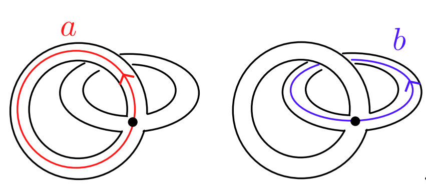
Example 16. The torus can be thought of as a square with opposite sides identified:



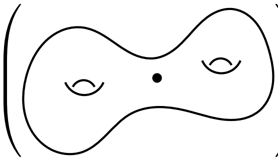
Therefore



The last space can easily collapsed onto two circles, so it turns out that the fundamental group is isomorpic to $\mathbb{Z} * \mathbb{Z}$ with generators

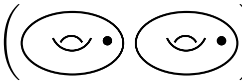


Tracing out the boundary shows that the boundary is homotopic to $aba^{-1}b^{-1}$. Hence

π_1 () $\cong \langle a, b\rangle * \langle c, d\rangle/(aba^{-1}b^{-1} = cdc^{-1}d^{-1})$

$$= \langle a, b, c, d | aba^{-1}b^{-1}c^{-1}d^{-1}cd = e\rangle.$$

The fundamental group is great because it is a group. It is not so great because it only sees one path component of X. You want to be able to talk about

π_1 () =???.

To talk about multiple basepoints, it is necessary to introduce the notion of a category.

Definition 17. A category $\mathcal{C}$ is

- a collection of objects $\mathrm{Obj}(\mathcal{C})$ and collections of morphisms $\mathrm{Mor}(x, y)$ for each $x, y \in \mathrm{Obj}(\mathcal{C})$
- associative composition maps

$$\circ : \mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z)$$

- identity morphisms $\mathrm{id}_x \in \mathrm{Mor}(x, x)$ such that id_x acts as the identity for composition

A category can be thought of as a collection of dots and arrows, where if two arrows lie end to end their composition is also an arrow.

Example 18. Fix a group G. Define the category $*/\!\!/G$ to be the category with a single object $*$, $\mathrm{Mor}(*, *) = G$, and composition given by group multiplication.

Example 19. More generally, let S be a set acted on by G (on the right, say). Define a category $S/\!\!/G$ by $\mathrm{Obj}(S/\!\!/G) = S$ and a morphism from s_0 to s_1 if $s_1 = s_0 g$.

Example 20. Let X be a topological space and S a nonempty subset of X. Let $\pi_1(X; S)$ be the category with objects S and $\mathrm{Mor}(x, y)$ the homotopy classes (rel endpoints, as usual) of curves from x to y.

For example, if $S = \{x\}$ is a single point, then $\pi_1(X; \{x\})$ is isomorphic[1] to $*/\!\!/\pi_1(X, x)$. After all, they each have one object and one morphism for every homotopy of class of loop based at x.

I will call $\pi_1(X; S)$ the fundamental groupoid of X with respect to S. Typically S will be a finite set. Officially, the phrase "fundamental groupoid" is reserved for $\pi_1(X; X)$. The word groupoid comes from

Definition 21. A groupoid is a category where every morphism is invertible.

[1] I have not defined what is meant by isomorphism here but the interested reader may investigate.

Note that in a groupoid all the sets $\mathrm{Mor}(x, x)$ are all isomorphic groups.

Just as there are morphisms between groups that preserve the structure (composition, identity), there are morphisms between categories that preserve the structure (composition, identity). For whatever reason, they're not called "category homomorphisms" but rather called "functors":

Definition 22. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is a map

$$F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}')$$

and a map

$$F : \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y)).$$

(Note the same letter F is used in two different ways. This is standard.) F must satisfy that

$$F(f \circ g) = F(f) \circ F(g)$$
$$F(\mathrm{id}_x) = \mathrm{id}_{F(x)}\,.$$

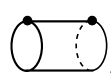
Example 23. The functors $*/\!\!/G \to */\!\!/H$ are essentially the same as group homomorphisms $G \to H$.

Definition 24. Let $\mathrm{Hom}(\pi_1(X; S), G)$ be the set of functors $\pi_1(X; S) \to */\!\!/G$.

Intuitively, $\mathrm{Hom}(\pi_1(X; S), G)$ is the assignment of a group element $g \in G$ to each curve in X in a way invariant under homotopy and consistent with composition. By the last example, $\mathrm{Hom}(\pi_1(X; \{x\}), G)$ is essentially the same thing as $\mathrm{Hom}(\pi_1(X, x), G)$.

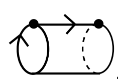
Example 25. $\mathrm{Hom}(\pi_1(T^2), G)$ is in bijection with pairs of commuting elements in G. A bijection is set by picking a pair of generators of $\pi_1(T^2)$.

Example 26. Consider the cylinder a basepoint on each end:



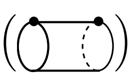
.

The following two arrowed curves generate the fundamental groupoid with respect to these points

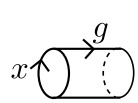


.

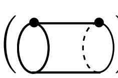
Hence

$$\mathrm{Hom}\left(\pi_1\left(\right.\right.$$  $$\left.\left.\right), G\right)$$

is in bijection with assignments of $(x, g) \in G^2$ to these two curves:



and

$$\mathbb{C}\,\mathrm{Hom}\left(\pi_1\left(\right.\right.$$  $$\left.\left.\right), G\right) \cong D(G)$$

as vector spaces.

Example 27. Just as there is a group algebra there is a groupoid algebra that has a basis given by the arrows of the groupoid. Composition of basis elements in the algebra is 0 if composition is not defined in the groupoid. You can check that $D(G)$ is isomorphic to the groupoid algebra of $G/\!\!/G$ where G acts on itself by conjugation.

Claim 28. *Let S' be obtained from S by adding a new point $s' \in X$ that lies in the same path component of some $s \in S$. Then*

$$\operatorname{Hom}(\pi_1(X; S'), G) \cong G \times \operatorname{Hom}(\pi_1(X; S), G).$$

Proof. Let γ be a fixed path from s to s'. Any hom from $\pi_1(X; S')$ to G is determined by its restriction to $\pi_1(X; S)$ and the group element to which the morphism $[\gamma]$ is sent. $\square$

Corollary 29. *Let X be path-connected and let $S \subset X$ be a finite subset. Then the quantity*

$$\frac{|\operatorname{Hom}(\pi_1(X; S), G)|}{|G|^{|S|}}$$

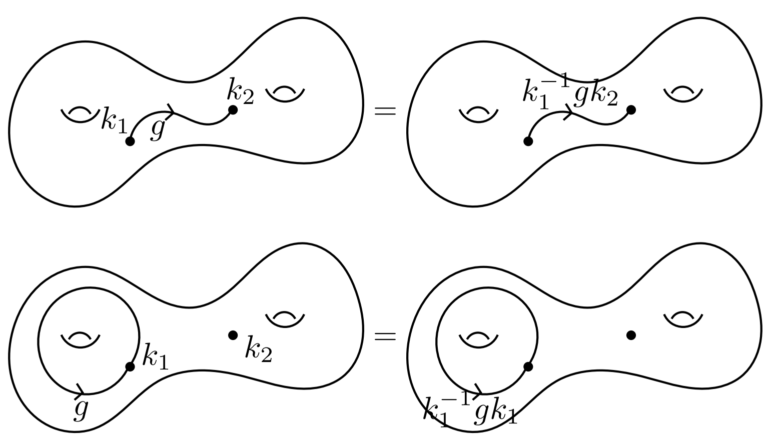
does not depend on the set S. In particular, it is always equal to

$$\frac{|\operatorname{Hom}(\pi_1(X), G)|}{|G|}.$$

Write G^S to denote functions $S \to G$. Define a G^S action on $\operatorname{Hom}(\pi_1(X; S), G)$ (the "gauge action") by

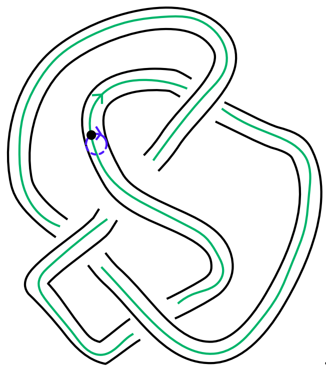
$$\phi \cdot f = f(\gamma(0))^{-1}\phi(\gamma)f(\gamma(1)), \ \phi \in \operatorname{Hom}(\pi_1(X; S), G), \ f \in G^S.$$

In pictures, here's an action of $(k_1, k_2) \in G^2$:



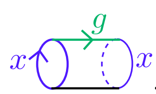
When S is a single point, this is the conjugation action on the set $\operatorname{Hom}(\pi_1 X, G)$. Let $s_0 \in S$. Claim 28 can be rephrased as: $G^{\{s_0\}}$ acts freely if s_0 shares a path component with another element of S.

Here is the idea behind quantum knot invariants. Remove a neighborhood of a knot in S^3:



.

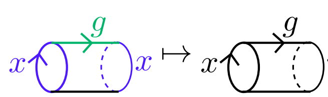
Write M for S^3 minus the neighborhood of the knot. As pictured, also pick a basepoint x in the boundary ∂M together with a choice of two directed curves: one bounds a disk in S^3 and the other runs once around the knot. An element in $\mathrm{Hom}(\pi_1(\partial M, x), G)$ is determined by the image of the blue and green curves. Let $x \in G$ be the image of the blue curve and $g \in G$ be the image of green curve. Then an element in $\mathrm{Hom}(\pi_1(\partial M, x), G)$ can be written



.

This provides a map

$$i : \mathbb{C}\,\mathrm{Hom}(\pi_1(\partial M, x), G) \to D(G).$$



.

For $\phi \in \mathrm{Hom}(\pi_1(M, x), G)$, write $\partial\phi$ to denote its restriction to ∂M. Then

$$I(K) = \sum_{\phi \in \mathrm{Hom}(\pi_1(M,x),G)} i(\partial\phi)$$

is an invariant of the knot (plus choice of green curve) that lives in $D(G)$. In fact, it lives in the center of $D(G)$ and so can be written[2]

$$I(K) = \sum_i a_i \pi_i$$

where π_i are the projectors of $D(G)$. Each coefficient a_i is also an invariant of the knot (plus choice of green curve). These are quantum link invariants.

Here, as always, G is a finite group. If it were to work for SU(2) (which it doesn't) then the Jones polynomial (evaluated suitably) would correspond to the coefficient of the 2d irrep of SU(2) (thought of as the stabilizer of the identity in SU(2)?).

One way to construct topological spaces is to glue together "simplices."

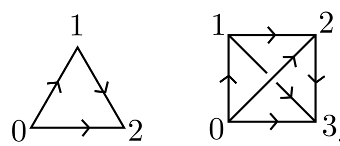
[2]Next week, this will be written in terms of the characters χ_i instead of the projectors π_i (because of the way I choose to orient things).

Definition 30. Suppose $x_0, \ldots, x_n$ is an ordered set of $n+1$ points in $\mathbb{R}^n$ that do not lie a lower-dimensional affine subspace. Their convex hull

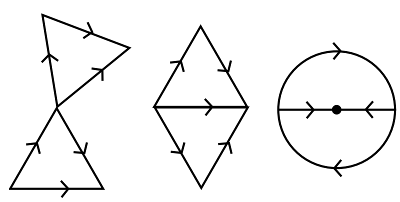
$$\left\{ \sum_{i=0}^{n} t_i x_i \,\middle|\, \sum_{i=0}^{n} t_i = 1 \right\} \subset \mathbb{R}^n$$

is called an n-simplex. There is a unique affine transformation taking one n-simplex to another preserving the vertex ordering. Hence all such simplices will be identified and called "the" n-simplex. It is denoted Δ^n. Note that $I = \Delta^1$.

The n-simplex is often drawn with arrows on its edges indicating the order of the vertices: the arrows point forward in the ordering of the vertices. Here is a 2-simplex and a 3-simplex.



A Δ-complex is a space built out of gluing together simplices along subsimplices using affine maps in a manner respecting the vertex ordering. For us, all Δ-complexes will contain finitely many simplices. For more technical details on Δ-complexs see Hatcher section 2.1. Here are some Δ-complexes:

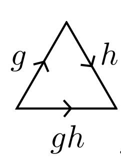


A Δ-complex structure on a space X is homeomorphism from X to a Δ-complex. In practice this is a decomposition of X into simplices with arrows on their edges.

Let K be a Δ-complex. Write $|K|$ for the underlying topological space. Write K^i for the restriction to all k simplices for $k \leq i$. In particular, K^0 is the set of vertices.

It is a fact that any curve with endpoints in K^0 can be homotoped rel endpoints to a curve lying in K^1. Hence

Claim 31. $\mathrm{Hom}(\pi_1(|K|, K^0), G)$ *is in bijection with labelings of the edges of* K *such that any 2-simplex is labeled like*



For simplicity, write

$$\mathrm{Hom}(K, G) := \mathrm{Hom}(\pi_1(|K|, K^0), G).$$

Definition 32. A combinatorial equivalence $f : K \to K'$ is a homeomorphism that takes simplices to simplices and preserves the orderings of the vertices.

Exercises:

1. Compute a presentation for the fundamental group of a surface of genus g.
2. Show how the multiplication in $D(G)$ comes from considering $\mathrm{Hom}(\pi_1(\Delta^2 \times S^1; S), G)$ for a S a certain subset of three points on the boundary of $\Delta^2 \times S^1$.
3. Let G be a finite group.
   (a) It is true that
   $$\sum_{g,h} ghg^{-1}h^{-1} = \sum_i \left(\frac{|G|}{\dim V_i}\right)^2 \pi_i$$
   Explain how the coefficients on the right are like quantum link invariants for a one-holed torus.
   (b) (hard) Prove the formula (hint: $\sum_{g,h} ghg^{-1}h^{-1} = \sum_\alpha |C_\alpha| \underline{\alpha} \cdot \underline{\alpha^{-1}}$ where $\underline{\alpha} = \sum_{g \in \alpha} g$ and α are the conjugacy classes of G).