Given vector spaces $V$ and $W$ over $k$ one can define a new vector space $V \otimes W$ called the tensor product of $V$ and $W$. The precise definition will come shortly, but $V \otimes W$ can be described in terms of a basis. Let $\{e_i\}_{i \in \mathcal{I}}$ be a basis for $V$ and $\{f_j\}_{j \in \mathcal{J}}$ a basis for $W$. Then $V \otimes W$ has a basis $\{e_i \otimes f_j\}_{i \in \mathcal{I},\ j \in \mathcal{J}}$. Here $e_i \otimes f_j$ is just a formal symbol. $V \otimes W$ then consists of elements of the form

$$\sum_{i,j} a_{ij} e_i \otimes f_j, \ a_{ij} \in k.$$

In particular, if $V$ and $W$ are finite dimensional, then $V \otimes W$ has dimension $\dim(V)\dim(W)$. Whereas $V \oplus W$ is a "sum" of vector spaces, $V \otimes W$ is the corresponding "product" of vector spaces.

**Definition 1.** Given vector spaces $V, W$ over a field $k$ let $\mathcal{V}$ be the free vector space on the set $V \times W$. Let $\mathcal{I}$ be the subspace spanned by

$$a(v,w) - (av,w), \ a \in k, \ v \in V, \ w \in W$$

$$a(v,w) - (v,aw), \ a \in k, \ v \in V, \ w \in W$$
$$(v_1 + v_1, w) - (v_1, w) - (v_2, w), \ v_i \in V, \ w \in W$$
$$(v, w_1 + w_2) - (v, w_1) - (v, w_2), \ v \in V, \ w_i \in W.$$

$V \otimes W$ is defined to be $\mathcal{V}/\mathcal{I}$. The class of $(v,w)$ in the quotient is denoted $v \otimes w$.

It is possible to recover the basis description of $V \otimes W$ from this definition. Let $\{e_i\}$ be a basis for $V$ and $\{f_j\}$ a basis for $W$. Then any element in $\mathcal{V}$ can be written as a (finite) sum

$$\sum_{v,w} a_{v,w}(v,w) = \sum_{v,w} a_{v,w} \left( \sum_i b^i e_i, \sum_j c^j f_j \right) \equiv \sum_{v,w} \sum_{i,j} a_{v,w} b^i c^j (e_i, f_j)$$

where $(e_i, f_j)$ is denoted $e_i \otimes f_j$.

**Remark 2.** It is not true that every element in $V \otimes W$ is of the form $v \otimes w$.

**Remark 3.** $V \otimes k$ consists of elements of the form $v \otimes 1$ for $v \in V$, $1 \in k$. Hence $V \otimes k$ can be canonically identified with $V$.

Write $\langle v, \xi \rangle$ to denote the pairing of a vector $v \in V$ with a dual vector $\xi \in V^*$.

**Proposition 4.** *Let $V$ and $W$ be finite dimensional vector spaces. There is a natural isomorphism*

$$V^* \otimes W \to \operatorname{Hom}(V, W)$$

$$\xi \otimes w \mapsto \langle v, \xi \rangle w.$$

*Proof.* These two spaces have the same dimension. The reader can check either injectivity or surjectivity. □

Let $\{e_i\}$ be a basis for $V$ with dual basis $\{e^i\}$. Let $\{f_j\}$ be a basis for $W$. Then any element in $\mathrm{Hom}(V, W)$ can be written

$$\sum_{i,j} a_i^j e^i \otimes f_j.$$

Here $a_i^j$ are the matrix entries for the linear transformation in terms of these bases. Note that matrix multiplication corresponds to pairing a vector space with its dual: for example, let $A \in \mathrm{End}(V) = \mathrm{Hom}(V, V)$ and $B \in \mathrm{End}(V)$ and write

$$A = \sum_{i,j} a_i^j e^i \otimes e_j$$

$$B = \sum_{k,l} b_k^l e^k \otimes e_l$$

so that

$$AB = \sum_{i,j,k,l} a_i^j b_k^l e^i \otimes \langle e_j, e^k \rangle \otimes e_l = \sum_{i,j,k,l} a_i^j b_k^l \delta_j^k e^i \otimes e_k = \sum_{i,k,l} a_i^k b_k^l e^i \otimes e_l.$$

Here I am using the canonical identification of $V \otimes k$ with $V$.

The identity map in $\mathrm{End}(V)$ is given by

$$\mathrm{id}_V = \sum_i e^i \otimes e_i.$$

In particular this shows that the expression on the right does not depend on the basis $\{e_i\}$. Therefore, the expression

$$\sum_i e_i \otimes e^i \in V \otimes V^*$$

also does not depend on the choice of basis. Thought of as an element of $(V^* \otimes V)^* = \mathrm{End}(V)^*$ this is the trace of the endomorphism. Thus the $\mathrm{id}_V$ and $\mathrm{tr} : \mathrm{End}(V) \to k$ are, in a certain sense, dual.

**Definition 5.** Given $A : V_1 \to V_2$ and $B : W_1 \to W_2$, define

$$A \otimes B : V_1 \otimes W_1 \to V_2 \otimes W_2$$

$$v \otimes w \mapsto (Av) \otimes (Bw).$$

**Definition 6.** Given $V$ and $W$ finite dimensional vector spaces and a linear map $A : V \to W$, define $A^* : W^* \to V^*$ by

$$\langle v, \eta A^* \rangle = \langle vA, \eta \rangle, \; v \in V, \eta \in W^*.$$

This is well-defined because in a finite dimensional vector space a dual vector is determined by how it pairs with vectors. $A^*$ is called the adjoint of $A$.

**Remark 7.** If

$$A = \sum_{i,j} a_i^j e^i \otimes f_j$$

then

$$A^* = \sum_{i,j} a_i^j f_j \otimes e^i$$

after identifying $W$ with $W^{**}$. It follows that $\mathrm{tr}(A) = \mathrm{tr}(A^*)$.

**Remark 8.** Since

$$\langle v, \eta(AB)^* \rangle = \langle vAB, \eta \rangle = \langle vA, \eta B^* \rangle = \langle v, \eta B^* A^* \rangle$$

it follows that $(AB)^* = B^* A^*$.

What follows are several constructions of representations of $G$:

**Example 9.** Let $\rho_V : G \to \mathrm{Aut}(V)$ be a representation. It will be useful to give $V^*$ a $G$-action as well. An obvious choice would be to define

$$g \mapsto \rho_V(g)^* \in \mathrm{Aut}(V^*)$$

but this is not a group homomorphism:

$$gh \mapsto \rho_V(gh)^* = (\rho_V(g)\rho_V(h))^* = \rho_V(h)^* \rho_V(g)^*.$$

To make it a group homomorphism, instead define the action by

$$g \mapsto \rho_V(g^{-1})^* \in \mathrm{Aut}(V^*).$$

This defines a representation

$$\rho_{V^*} : G \to \mathrm{Aut}(V^*).$$

Since $\mathrm{tr}(A) = \mathrm{tr}(A^*)$, then

$$\chi_{V^*}(g) = \chi_V(g^{-1}).$$

When $G$ is a finite group and $V$ is a complex representation, all the eigenvalues of $\rho_V(g)$ are finite order and hence roots of unity. Since the inverse of a root of unity is its complex conjugate, it follows that

$$\chi_{V^*}(g) = \overline{\chi_V(g)}.$$

**Example 10.** Let $V, W$ be representations of $G$. Define a representation of $G$ on $V \otimes W$ by

$$\rho_{V \otimes W} : G \to \mathrm{Aut}(V \otimes W)$$
$$g \mapsto \rho_V(g) \otimes \rho_W(g).$$

Since $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$, it follows that

$$\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g).$$

**Example 11.** Any group has a "trivial representation" on the ground field $k$

$$\rho_0 : G \to \mathrm{Aut}(k)$$

$$g \mapsto 1.$$

Let $V_0$ denote $k$ with this action. Note that

$$\chi_0(g) = 1, \ \forall g \in G.$$

**Example 12.** $\mathrm{Hom}(V, W)$ admits a $G$-action by

$$\rho_{\mathrm{Hom}(V,W)} : G \to \mathrm{Aut}(\mathrm{Hom}(V, W))$$

$$A \cdot g = \rho_V(g^{-1}) A \rho_W(g), \ A \in \mathrm{Hom}(V, W).$$

It is left as an exercise to show that the canonical isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ gives an equivalence the representations $V^* \otimes W$ and $\mathrm{Hom}(V, W)$.

$\mathrm{Hom}_G(V, W)$ is the subset of linear maps that commute with the $G$-action, i.e., $A \in \mathrm{Hom}_G(V, W)$ if $A\rho_W(g) = \rho_V(g)A$, i.e., $\rho_V(g^{-1})A\rho_W(g) = A$. Hence $\mathrm{Hom}_G(V, W)$ is precisely the trivial part of $\mathrm{Hom}(V, W)$.

**Definition 13.** An algebra $A$ over a field $k$ is a $k$-vector space with a multiplication map

$$m : A \otimes A \to A.$$

The multiplication is typically denoted by placing $a$ and $b$ next to each other, that is, $m(a, b)$ is written like $ab$.

**Example 14.** The basic example of an algebra is the vector space of $k$-valued functions on a set $X$. Pointwise multiplication of functions $(fg)(x) = f(x)g(x)$ defines an algebra structure.

**Example 15.** $\mathrm{End}(V)$ is an algebra. Multiplication is given by composition of endomorphisms.

**Example 16.** Given algebras $A$ and $B$, $A \oplus B$ is an algebra with multiplication

$$(a, b)(a', b') = (aa', bb').$$

**Example 17.** Let $kG$ be the free $k$-vector space on the set $G$. In particular $kG$ has a basis $\{g\}_{g \in G}$. Put a multiplication on $kG$ by extending the group multiplication from the basis:

$$(ag + bh)(a'g' + b'h') = aa'gg' + ab'gh' + ba'hg' + bb'hh', \ a, a', b, b' \in k$$

**Definition 18.** An algebra homomorphism $\phi : A \to B$ is a linear map that commutes with the multiplication

$$\phi(aa') = \phi(a)\phi(a').$$

Just as a group acts on a set $X$ via a group homomorphism

$$G \to \mathrm{Aut}(X)$$

(where $\mathrm{Aut}(X)$ is the group of self-bijections of $X$) an algebra "acts" on a vector space $V$ via an algebra homomorphism

$$A \to \mathrm{End}(V).$$

**Definition 19.** A module for the $k$-algebra $A$ is a $k$-vector space $V$ together with a algebra homomorphism

$$\rho : A \to \mathrm{End}(V).$$

The notation $\rho$ is meant to evoke a group representation. In fact:

**Proposition 20.** *There is a bijection between representations of the group $G$ and modules for the algebra $kG$.*

*Proof.* Given a representation $\rho : G \to \mathrm{Aut}(V)$, extend this linearly to a map

$$kG \to \mathrm{End}(V)$$

by

$$\sum_g a_g g \mapsto \sum_g a_g \rho_V(g).$$

Conversely, an algebra homomorphism

$$kG \to \mathrm{End}(V)$$

restricts to $G \subset kG$ to a map

$$G \to \mathrm{End}(V).$$

Since every element of $G$ is invertible in $kG$, the image lands in $\mathrm{Aut}(V) \subset \mathrm{End}(V)$. □

**Definition 21.** An $A$-module is called simple if it $A$ does not send a proper nonzero subspace to itself.

**Remark 22.** In particular, irreducible $G$-representations over $k$ and in bijection with simple $kG$-modules.

The words $kG$-module and $G$-representation will be used interchangeably in what follows.

From now on, fix $k = \mathbb{C}$ and let $G$ be a finite group. One of the most important elements in $\mathbb{C}G$ is

$$\pi_0 := \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}G$$

called the "trivial projector":

**Claim 23.** *For $V$ a finite dimensional $\mathbb{C}G$-module, write*

$$V \cong \bigoplus_i V_i^{n_i}.$$

*As usual write $V_0$ for the trivial representation. Then $\pi_0$ projects $V$ onto the $V_0^{n_0}$ part.*

*Proof.* Since each summand is a submodule of $V$, $\pi_0$ sends it to itself. Since

$$v\left(\frac{1}{|G|}\sum_g g\right)h = v\left(\frac{1}{|G|}\sum_g g\right)$$

it follows that the image of $\rho_V(\pi_0)$ is the trivial part. It is not hard to check that $\pi_0^2 = \pi_0$ in $\mathbb{C}G$. □

Let $\{V_i\}$ be a collection of irreducible complex $G$-representations, (equivalently, a collection of simple $\mathbb{C}G$-modules) with precisely one from each equivalence class of representation. Write $\chi_i$ for the character of $V_i$. $\chi_i$ does not depend on the precise choice of representation in that equivalence class.

A character $\chi_V$ is a function on $G$ but can extended linearly to give an element in $(\mathbb{C}G)^*$. Since the trace of a projection is the dimension of its image, it follows that $\chi_V(\pi_0)$ is the dimension of the trivial part of $V$.

Note that $\chi_V$ are "class functions", that is, they are constant on conjugacy classes of $g$. Define a Hermitian inner product on complex-valued functions on $G$

$$(\phi, \psi) := \frac{1}{|G|}\sum_g \overline{\phi(g)}\psi(g).$$

**Proposition 24.** *The characters of irreducibles, $\{\chi_i\}$, are orthonormal with respect to this inner product. In particular, they are linearly independent in the space of class functions and so there are finitely many equivalence classes of irreps.*

*Proof.* Note that the trivial part of $\mathrm{Hom}(V, W)$ is $\mathrm{Hom}_G(V, W)$, so Schur's lemma says that

$$\chi_{\mathrm{Hom}(V_i,V_j)}(\pi_0) = \begin{cases} 1 & V_i \cong V_j \\ 0 & V_i \ncong V_j \end{cases}.$$

On the other hand

$$\chi_{\mathrm{Hom}(V_i,V_j)}(\pi_0) = \frac{1}{|G|}\sum_g \chi_{\mathrm{Hom}(V_i,V_j)}(g) = \frac{1}{|G|}\sum_g \chi_{V_i^*\otimes V_j}(g)$$

$$= \frac{1}{|G|}\sum_g \chi_i(g^{-1})\chi_j(g) = \frac{1}{|G|}\sum_g \overline{\chi_i(g)}\chi_j(g) = (\chi_i, \chi_j).$$

□

It is now possible to prove the important theorem from yesterday:

**Theorem 25.** *If $\chi_V = \chi_W$ then $V$ is equivalent to $W$.*

*Proof.* Write

$$V \cong \bigoplus_i V_i^{n_i}, \; W \cong \bigoplus_j V_j^{m_j}.$$

Then

$$\chi_V = \sum_i n_i \chi_i, \; \chi_W = \sum_j n_j \chi_j.$$

Since characters are linearly independent, the coefficients on the left and the right must be the same, and so $V_i$ appears the same number of times in each of $V$ and $W$. □

For each irrep $V_i$ there's a projector in $\mathbb{C}G$ that projects onto the $V_i$ part of any representation.

**Claim 26.** *The element*

$$\pi_i = \frac{\dim V_i}{|G|} \sum_g \chi_i(g^{-1}) g$$

*acts on any representation by projecting onto the $V_i$ part.*

*Proof.* See the exercises at the end. □

$\mathbb{C}G$ acts on itself on the right (say) and so becomes a right $\mathbb{C}G$-module of dimension $|G|$. Each $g$ acts as a permutation matrix whose diagonal entries are all zero unless $g = e$. Hence

$$\chi_{\mathbb{C}G}(g) = \begin{cases} |G| & g = e \\ 0 & \text{otherwise} \end{cases}.$$

The number of times that $V_i$ appears in $\mathbb{C}G$ is given by

$$\frac{1}{\dim V_i} \chi_{\mathbb{C}G}(\pi_i) = \frac{1}{|G|} \sum_g \chi_i(g^{-1}) \chi_i(g) = \dim V_i$$

hence $V_i$ appears in $\mathbb{C}G$ precisely $V_i$ times. In particular,

$$|G| = \sum_i (\dim V_i)^2.$$

More is true, however,

**Proposition 27.** *The algebra homomorphism*

$$\mathbb{C}G \to \bigoplus_i \operatorname{End}(V_i)$$

$$g \mapsto \sum_i \rho_i(g)$$

*is an isomorphism.*

*Proof.* Since the two sides have the same dimension, it is enough to show injectivity. Suppose

$$\sum_g a_g g$$

is in the kernel. Then it acts as zero on all irreducible representations and so by Maschke's theorem acts as zero on all representations. Then it acts as zero as right multiplication on $\mathbb{C}G$. Then

$$e\left(\sum_g a_g g\right) = 0.$$

This implies that $a_g = 0$ for all $g$. □

**Corollary 28.** *The projectors* $\{\pi_i\}$ *span the center of* $\mathbb{C}G$*.*

*Proof.* The center of $\mathrm{End}(V_i)$ are the scalar multiplies of the identity on $V_i$. The center of a sum of algebras is the sum of the centers. □

**Corollary 29.** *The number of equivalence classes of irreps is the number of conjugacy classes of* $G$*.*

*Proof.* The center of $\mathbb{C}G$ is the trivial part under the action by conjugation:

$$g \cdot h = h^{-1}gh.$$

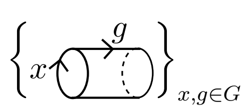
The dimension of this trivial part is the number of conjugacy classes. □

**Corollary 30.** *The characters* $\{\chi_i\}$ *form a basis for class functions on* $G$*.*

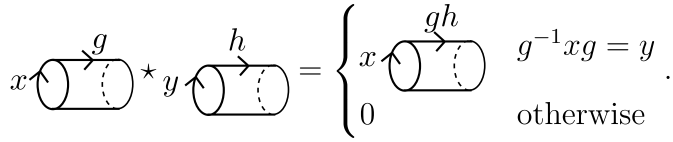
*Proof.* The set $\{\chi_i\}$ is already linearly independent and the dimension of the space of class functions if the number of conjugacy classes of $G$. □

I will now define a $\mathbb{C}$-algebra $D(G)$ that is very much like $\mathbb{C}G$.

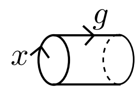
**Definition 31.** $D(G)$ is the $\mathbb{C}$-vector with basis given by symbols



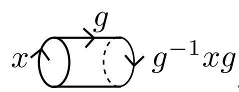

and multiplication



The multiplication is topologically motivated. The full story will come later, but for now consider the symbol




also a labeled by $g^{-1}xg$ on the right:




This is a cylinder with three curves labeled by group elements. Multiplication is essentially concatenation of these labeled cylinders.

The labels on the two circles are called the "meridian labels" and the label on the top is called the "longitude label."

Exercises:

1. Let $G = S_3$ and $(\rho_2, V_2)$ the 2-dimensional irrep. What is the decomposition of $V_2 \otimes V_2$? (Harder: can you come up with a decomposition of $V_2^{\otimes n}$?)

2. $G = S_3$, generated by $r$ and $s$ as yesterday. Over the complex numbers, $\rho_2$ becomes conjugate to

$$r \mapsto \begin{pmatrix} \zeta & \\ & \zeta^2 \end{pmatrix}, \; \zeta = e^{\frac{2\pi i}{3}}$$

$$s \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

   Using the tensor product of this basis, find bases for the components of $V_2 \otimes V_2$.

3. $G = S_3$. What is the relation between $P$ yesterday and $V_2 \otimes \mathbb{R}^3$, where $S_3$ acts on $\mathbb{R}^3$ by permuting the basis vectors?

4. Over $\mathbb{C}$, Schur's lemma says that if $V$ and $W$ are irreps of $G$, then

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & V \cong W \\ 0 & \text{otherwise} \end{cases}.$$

   The "proof" is that, for $A \in \mathrm{Hom}_G(V, W)$, $\ker(A)$ and $\mathrm{im}(A)$ are subrepresentations. Turn this "proof" into an actual proof.

5. Show that

$$\pi_i := \frac{\dim V_i}{|G|} \sum_g \chi_i(g^{-1}) g$$

   is a projector onto the $i$th irrep. The proof I have in mind goes like 1) show that $g\pi_i g^{-1} = \pi_i$, 2) show that $\rho_{V_j}(\pi_i)$ is multiplication by a constant, 3) compute that constant by taking a trace.

6. Schur's lemma fails over $\mathbb{R}$. Can you find an example of a group where it fails? Does the group algebra decompose a sum of matrix algebras in this case?

7. For $\alpha$ a conjugacy class in the group, write

$$\sum_{g \in \alpha} \delta_g \in (\mathbb{C}G)^*.$$

in terms of the (complex) characters of $G$.

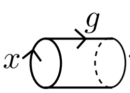
8. Show, over $\mathbb{C}$, irreps of abelian groups are 1-dimensional. Can you comment on the relation between this and the Fourier transform $L^2(S^1) \leftrightarrow \ell^2(\mathbb{Z})$?

9. Prove that every element in the center of $D(G)$ is a sum of basis elements where the left and right meridian labels are the same.

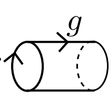
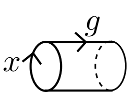
10. Define an embedding

$$i : \mathbb{C}G \to D(G)$$

by

$g \mapsto \sum_x x$  .

Let $X \subset D(G)$ be the subspace spanned by the basis elements where the left and right labels are the same. Let $G$ act on $X$ via conjugation:

$x$  $\cdot\, h := i(h^{-1}) \star x$  $\star\, i(h).$

Show that the trivial part of this action is the center of $D(G)$.