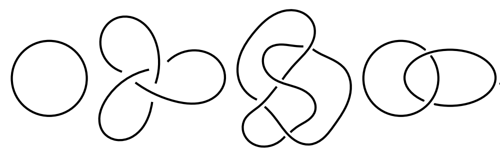
“Quantum invariants” of 3-manifolds are so named because they were first announced in Ed Witten’s paper “Quantum Field Theory and the Jones Polynomial.” As the title of the paper indicates, they have a lineage back to a particular polynomial discovered by Vaughan Jones when he (Jones) was studying von Neumann algebras.

A link in $\mathbb{R}^3$ is an embedding of some copies of the circle in $\mathbb{R}^3$:

$$S^1 \sqcup \cdots \sqcup S^1 \hookrightarrow \mathbb{R}^3.$$

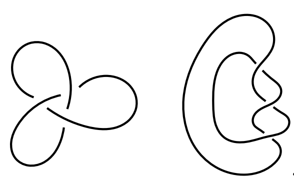
If a link has a single component it is called a knot. A link with a chosen direction around each circle is called an oriented link. Here are some pictures, three knots and a two component link:



Two links are equivalent if one can be deformed to the other without passing through itself. It is true that the first three knots are all distinct, and this should be clear after a little fiddling—but to prove they are distinct can be thorny task. The last link is obviously distinct from the first three since it has two components while the first three only have one.

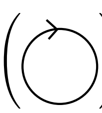
The Jones polynomial is an assignment of a “polynomial” (really an element of $\mathbb{Z}[q^{\pm 1/2}]$) to each equivalence class of oriented links. If two oriented links have different Jones polynomials, then they are not equivalent (though the converse does not hold).

A link is a 3-dimensional object since it sits in $\mathbb{R}^3$. A link diagram is a picture of that link drawn in $\mathbb{R}^2$ that indicates under and over crossings. For example, here are two distinct diagrams of equivalent links:

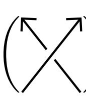
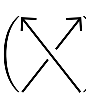
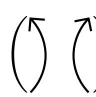


The Jones polynomial J is defined by the following algorithm applied to an oriented link diagram. It is a nontrivial theorem (due to Jones) that the algorithm leads to the same result if you use two different diagrams of equivalent links.

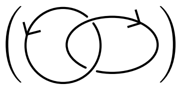
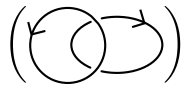
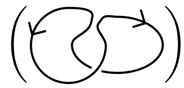
If the diagram is of the unknot, assign it 1:

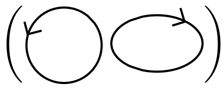
$J\Big($  $\Big) = 1.$

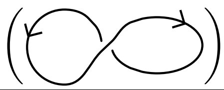
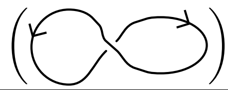
If three link diagrams differ at just a single crossing, then their Jones polynomials satisfy the relation

$qJ\Big($  $\Big) - q^{-1}J\Big($  $\Big) = (q^{1/2} - q^{-1/2})J\Big($  $\Big).$

This is best seen through an example:

J  $= q^{-1}\Big[q^{-1}J$  $+ (q^{1/2} - q^{-1/2})J$  $\Big]$

$= q^{-2}J$  $+ q^{-1}(q^{1/2} - q^{-1/2})$

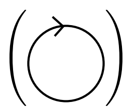
$= q^{-2}\Bigg[\Big(qJ$  $- q^{-1}J$  $\Big) \Big/ (q^{1/2} - q^{-1/2})\Bigg] + q^{-1/2} - q^{-3/2}$

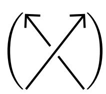
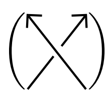
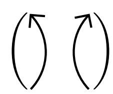
$$= q^{-2}\left[\frac{q - q^{-1}}{q^{1/2} - q^{-1/2}}\right] + q^{-1/2} - q^{-3/2}$$

$$= q^{-1/2} + q^{-5/2}$$

The computations during the course of running the algorithm depend on the diagram you pick for the link, but the end result does not depend on the choice of diagram. In a perfect world, there should be a definition of the Jones polynomial that does not depend on particular 2d projection of the link. It is somewhat vexing that no such definition has been discovered.

There is another link polynomial, the Alexander polynomial, given by

A  $= 1.$

A  $- A$  $= (q^{1/2} - q^{-1/2})A$ 

that admits several descriptions that do not depend on the 2-dimensional depiction of the link. Given how similar A and J are, you would think that there would be similar descriptions of the Jones polynomial, but none have been found.

In "Quantum Field Theory and the Jones Polynomial", Witten proposed an inherently 3-dimensional construction of the Jones polynomial[1]:

$$J(K)|_{q=e^{\frac{2\pi i}{k+2}}} \stackrel{?}{=} \int_{\mathcal{A}} DA \exp(ik\mathrm{CS}(A))\, \mathrm{tr}_{V_1}(\mathrm{hol}_K(A)),\ \ k \in \mathbb{Z}_{>0}.$$

I will not elaborate on this integral only to say that the space $\mathcal{A}$ is infinite-dimensional and the measure DA has not been constructed. Numerical evidence suggests that DA should exist. Witten knew enough about its predicted properties that he was able to confirm this equation on the physical level of rigor.

Witten's integral expression for the Jones polynomial fits into a larger class of invariants of 3-manifolds and links in those 3-manifolds. These are the so-called "quantum invariants." They depends on a compact Lie group G. The

[1]this expression has to be suitably normalized

Jones polynomial is the link invariant corresponding to $G = \mathrm{SU}(2)$. It is perhaps a little surprising that the Jones polynomial should have anything to do with SU(2).

Witten did not construct these invarants rigorously. He only predicted what properties they should have. Reshetikhin and Turaev, shortly after Witten's paper, constructed a family of invariants that exactly agreed with Witten's predictions.

Reshetikhin and Turaev's work does not depend at all on a path integral. To compress a long story, it, like Jones's original work, relies on applying an algorithm to a link diagram and then checking that the result does not depend on the original choice of diagram. The curious part about Reshetikhin and Turaev's work is that they produce their invariants from representations of a certain algebra $U_q\mathfrak{g}$. Here $\mathfrak{g}$ is the complexification of the Lie algebra of Witten's compact group. I won't write down what $U_q\mathfrak{g}$ is, but it has no immediate relation to topology. It is remarkable that one can extract from it link invariants.

Let G be a finite group. The protagonist of these lectures will be an algebra $D(G)$ from which one can also extract link invariants. Moreover, these link invariants also come from an integral like Witten's—the integral turns out to be an integral over a finite set, so a finite sum.

The finite group link invariants from $D(G)$ (to be discussed next week) are principally from work of Dijkgraaf and Witten ("Topological Gauge Theories and Group Cohomology"), Freed ("Higher Algebraic Structures and Quantization"), and Altschuler and Coste ("Quasi-Quantum Groups, Knots, Three-manifolds, and Topological Quantum Field Theory").

The link invariants will depend on the representation theory of $D(G)$. It is best, however, to start with the representation theory of the finite group G. A reference is the first two sections of Fulton and Harris.

Definition 1. A representation of a group G is a homomorphism $\rho : G \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ is the group of linear automorphisms of a vector space.

In particular, a representation provides an action of the group G on the vector space V. In notating actions, one has to make a choice to use left or right action. *I will use right actions unless stated otherwise.* This is contrary to the usual preference for left actions, but will make life easier for topological applications. A consequence of this convention is that linear maps are represented by matrices acting on row vectors.

Example 2. Write V_2 for the real plane, and fix an equilateral triangle centered at 0. Let G_0 be the group of isometries of the plane that fix this equilateral triangle. Then $G_0 \cong S_3$, an explicit isomorphism coming from the action of G_0 on the vertices.

Since isometries that fix the triangle are linear, there is by definition and inclusion

$$\rho_2 : G_0 \hookrightarrow \mathrm{Aut}(V_2)$$

that defines a representation.

Here are two other representations

$$\rho_1 : G_0 \to \mathrm{Aut}(\mathbb{R})$$

$$\rho_1(g) := \det(\rho_2(g))$$

and

$$\rho_0 : G_0 \to \mathrm{Aut}(\mathbb{R})$$

$$\rho_0(g) := (\rho_1(g))^2.$$

Write $V_1 = \mathbb{R}$ for the copy of $\mathbb{R}$ that G acts on through ρ_1 and $V_0 = \mathbb{R}$ for the copy of $\mathbb{R}$ that G acts on through ρ_0.

The word "representation" is a bit of a misnomer. A representation need not be injective, as these examples show.

The group G_0 is generated by two elements: a rotation by $2\pi/3$, call it r, and a reflection in a line, call it s. Pick a basis for V_2 such that the equilateral triangle has vertices at $(1, 0)$ and $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. Suppose that s fixes the line through $(1, 0)$. This choice of basis provides an isomorphism $A : V \to \mathbb{R}^2$ and so induces a new representation

$$\rho_2' : G_0 \to \mathrm{GL}_2\mathbb{R} = \mathrm{Aut}(\mathbb{R}^2)$$

defined by

$$\rho_2'(g) := A^{-1}\rho_2(g)A.$$

Recall that the matrix $\rho_2'(g)$ acts on the right. Then (up to fixing a choice of whether r rotates clockwise or counterclockwise)

$$\rho_2'(r) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ \rho_2'(s) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Since s and r generate G_0, these two matrices determine the entire representation. One can also compute

$$\rho_1(r) = 1, \ \rho_1(s) = -1$$

$$\rho_0(r) = 1, \ \rho_0(s) = 1.$$

The basis that defines ρ_2' is the "standard" basis for $\mathbb{R}^2$ but is not very symmetric for the action of G_0 on V_2. A better basis: let the first basis vector be that fixed by s and let the second basis vector be the first rotated by r. This new basis defines an isomorphism $B : V_2 \to \mathbb{R}^2$ and hence a new representation

$$\rho_2'' : G_0 \to \mathrm{GL}_2\mathbb{R}$$

$$\rho_2''(g) := B^{-1}\rho_2(g)B.$$

You can check that

$$\rho_2''(r) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\ \rho_2''(s) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

ρ_2, ρ_2', and ρ_2'' are all essentially they same, since you can from one to the other by conjgation by a fixed matrix.

Definition 3. Two representations $\rho_V : G \to \mathrm{Aut}(V)$ and $\rho_W : G \to \mathrm{Aut}(W)$ are equivalent if there exists a matrix $A : V \to W$ such that $\rho_W(g) = A^{-1}\rho_V(g)A$. This forms an equivalence relation on representations.

Given $\rho_V : G \to \mathrm{Aut}(V)$ and $\rho_W : G \to \mathrm{Aut}(W)$, G acts on $V \oplus W$ via

$$(v, w)g = (v\rho_V(g), w\rho_W(g))$$

(remember all actions are right actions). That is, there is a new representation

$$\rho_{V\oplus W} : G \to \mathrm{Aut}(V \oplus W)$$

where every element of G acts "block diagonally"

$$g \mapsto \begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix}$$

so the V subspace is sent to itself and the W subspace is sent to itself.

Definition 4. A representation is said to be "irreducible" if it does not send any proper nonzero subspace to itself.

Theorem 5 (Maschke)**.** *Let G be a finite group and $\rho_V : G \to \mathrm{Aut}(V)$ a representation over a field of characteristic 0. If V is not irreducible, then V is equivalent to a direct sum of two nonzero representations.*

Proof. Look up the proof in Fulton and Harris. □

Iterating this theorem, any finite-dimensional representation of a finite group is equivalent to a direct sum of irreducible representations. If someone hands you a representation, it is often in your interest to figure out this equivalence—for then the matrices of the representation are all block diagonal.

You can check that the G_0 representations ρ_0, ρ_1, and ρ_2 are all irreducible. In fact, it will turn out any irreducible representation is equivalent to one of these. In particular, any finite-dimensional representation V of G_0 will be equivalent to a direct sum

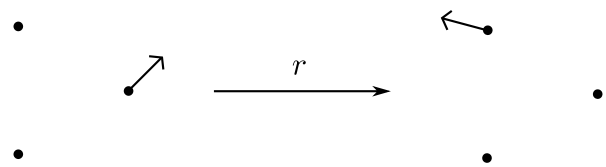
$$V \cong V_0^{n_0} \oplus V_1^{n_1} \oplus V_2^{n_2}$$

for some nonnegative integers n_i. Here is exhibited a common abuse of terminology: one writes the vector space V_i to denote a representation $\rho_i : G \to \mathrm{Aut}(V_i)$. It would perhaps be more precise to write something like

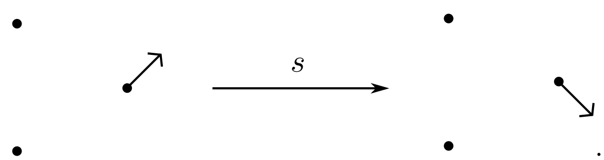
$$\rho_V \cong \rho_0^{n_0} \oplus \rho_1^{n_1} \oplus \rho_2^{n_2}$$

but this is not customary. At any rate, while people often write V to denote a representation, a representation really is a pair (ρ, V), where $\rho : G \to \mathrm{Aut}(V)$. In the same vein, the expression $V \cong W$ means (ρ_V, V) is equivalent to the representation (ρ_W, W).

Example 6. Let X be the vertex set of the equilateral triangle in V_2. For $x \in X$, the set of displacements from x forms a vector space with origin x. Moreover, r takes displacements around x and rotates them geometrically to displacements from xr



and s reflects them



Let W denote the 6-dimensional vector space of such displacements and let $P : G_0 \to \mathrm{Aut}(W)$ denote the action of G on the displacements. It turns out that (P, W) is equivalent to $V_0 \oplus V_1 \oplus V_2 \oplus V_2$, so that there exists a basis for W in which

$$P(r) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & -1 & -1 & & \\ & & & & & 1 \\ & & & & -1 & -1 \end{pmatrix}$$

$$P(s) = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & -1 & -1 & & \\ & & & & 1 & \\ & & & & -1 & -1 \end{pmatrix}.$$

It is an exercise at the end to find these bases.

Definition 7. Given $\rho_V : G \to \mathrm{Aut}(V)$ a representation over a field k, define $\chi_V : G \to k$ by

$$\chi_V(g) = \mathrm{tr}(\rho(g)).$$

χ_V is called the character of the representation V.

Of course, the notation really should be χ_{ρ_V} so as to indicate the representation, but it is common to just write the vector space in the subscript, χ_V.

Example 8. For G_0:

$$\chi_0(e) = 1,\ \chi_0(s) = 1,\ \chi_0(r) = 1$$
$$\chi_1(e) = 1,\ \chi_1(s) = -1,\ \chi_1(r) = 1$$
$$\chi_2(e) = 2,\ \chi_2(s) = 0,\ \chi_2(r) = -1.$$

The trace is invariant under conjugation and this has two consequences: 1) $\chi_V = \chi_W$ for V and W equivalent representations and 2) a character is constant on the conjugacy classes of G. For example, for G_0, a character is determined by its values of e, s, and r.

The "character table" of a group is table showing the values of all the characters of irreducibles. For example, the character table of G_0 is:

	$[e]$	$[s]$	$[r]$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

Note that $\chi_{V\oplus W} = \chi_V + \chi_W$. Since any representation is equivalent to a sum of irreducibles, any character is a sum of characters of irreducibles.

A representation is a lot of information: at a minimum, it's a linear transformation for each generator of the group. A character is, on the other hand, very simple: it's just a single function on the group. The following is therefore remarkable:

Theorem 9. *Let (ρ_V, V) and (ρ_W, W) be two representations. If $\chi_V = \chi_W$, then V is equivalent to W.*

This theorem will be proved tomorrow.

Let V and W be vector spaces over a field k. Write $\mathrm{Hom}(V, W)$ to denote k-linear maps from V to W. These are maps that commute with the action of k. If V and W are representations, write $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$ to denote those maps that also commute with the action of G. Formally:

Definition 10. Let (ρ_V, V) and (ρ_W, W) be two representations of G. The space $\mathrm{Hom}_G(V, W)$ consists of linear maps $F : V \to W$ such that $F(v\rho_V(g)) = F(v)\rho_W(g)$.

To be more pedantic, one should probably write $\mathrm{Hom}_G(\rho_V, \rho_W)$ since this space depends on the G-actions on V and W. Also note that the equation

$$F(v\rho_V(g)) = F(v)\rho_W(g)$$

can be written more concisely if ρ_V and ρ_W are understood:

$$F(vg) = F(v)g.$$

Remark 11. An isomorphism in $\mathrm{Hom}_G(V, W)$ is the same thing as a linear map $V \to W$ that conjugates the G-action on V to the G-action on W. Hence two representations V and W equivalent iff there exists an isomorphism in $\mathrm{Hom}_G(V, W)$.

Lemma 12 (Schur's Lemma)**.** *Let G be a finite group and V and W irreducible representations over $\mathbb{C}$. Then*

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & V \cong W \\ 0 & V \ncong W \end{cases}.$$

Proof. The image and kernel of an element in $\mathrm{Hom}_G(V, W)$ are subrepresentations. □

If V is irreducible, then Schur's lemma says that there are $\mathbb{C}$-many isomorphisms from V to itself. These therefore must be of the form $c\,\mathrm{id}_V$ for some $c \in \mathbb{C}$.

From now on, all representations will be over $\mathbb{C}$. Note that any real representation induces a complex representation (simply by including the real numbers into the complex numbers). In particular, ρ_0, ρ_1, and ρ_2 give complex complex representations of G_0.

Exercises:

1. Let V be the free vector space on the set $\{1, 2, 3\}$. S_3 acts on this set by permutations, and this action extends linearly to an action on V. How does this representation decompose? Can you come up with bases for the various components and the matrices for the action of S_3?

2. Same as problem 1, but for the representation P as done in class.

3. Let G be the isometries of $\mathbb{R}^3$ that preserve the regular tetrahedron in $\mathbb{R}^3$. Prove that G is isomorphic to S_4. Write down as much of the character table of S_4 as possible.

4. Over $\mathbb{C}$, Schur's lemma says that if V and W are irreps of G, then

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & V \cong W \\ 0 & \text{otherwise} \end{cases}.$$

   The "proof" is that, for $A \in \mathrm{Hom}_G(V, W)$, $\ker(A)$ and $\mathrm{im}(A)$ are subrepresentations. Turn this "proof" into an actual proof.