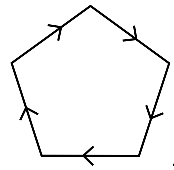
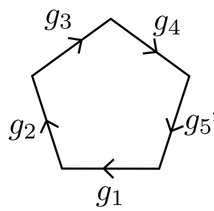
Direct the edges of a polygon either clockwise or counterclockwise, say



The G-labelings (representations of the fundamental groupoid with respect to the vertices) look like



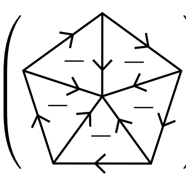
$$g_1 g_2 g_3 g_4 g_5 = e.$$

Encode this in the expression

$$\sum_{g_1 g_2 g_3 g_4 g_5 = e} \delta_{g_1} \otimes \delta_{g_2} \otimes \delta_{g_3} \otimes \delta_{g_4} \otimes \delta_{g_5}.$$

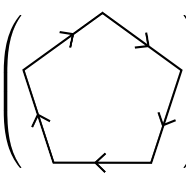
In the language of the 2D TQFT, this is

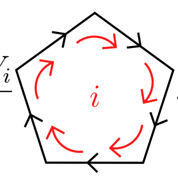
$$Z\left(\right)$$



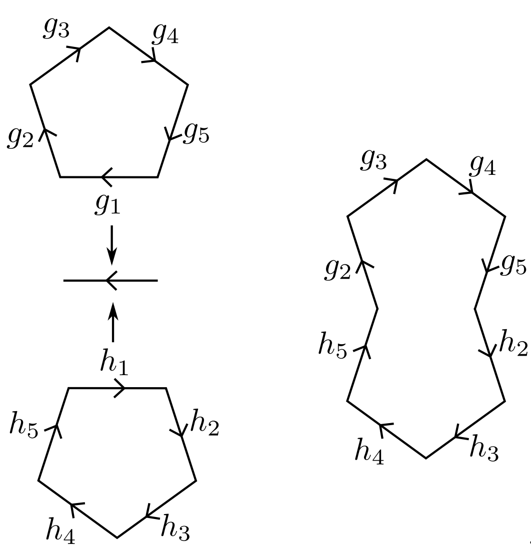
and hence can be drawn in the red arrow notation as

$$Z\left(\right) = \sum_i \frac{\dim V_i}{|G|}$$

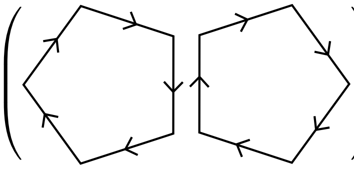




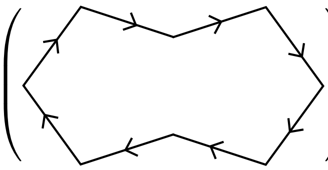
The G-labelings of an octagon can be obtained by gluing together G-labelings of two pentagons:



The G-labelings of the two disjoint pentagons are encoded by

$Z\Bigg($  $\Bigg) = \sum_{\substack{g_1g_2g_3g_4g_5=e \\ h_1h_2h_3h_4h_4=e}} \delta_{g_1}\otimes\cdots\delta_{g_5}\otimes\delta_{h_1}\otimes\cdots\otimes\delta_{h_5}.$

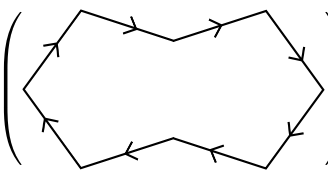
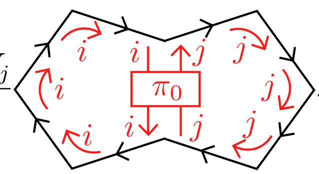
To get the corresponding for the G-labelings of the octagon, you need to sum over all the possibilities where $g_1 = h_1^{-1}$. Hence you need to contract the $\delta_{g_1}\otimes\delta_{h_1}$ part with $\sum_g g\otimes g^{-1}$:

$Z\Bigg($  $\Bigg) = \sum_{\substack{g_1g_2g_3g_4g_5=e \\ h_1h_2h_3h_4h_4=e}} \left\langle \sum_g g\otimes g^{-1}, \delta_{g_1}\otimes\delta_{h_1}\right\rangle \delta_{g_2}\otimes\cdots\otimes\delta_{g_5}\otimes\delta_{h_2}\cdots\otimes\delta_{h_5}$

Of course

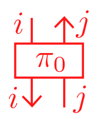
$$\sum_g g\otimes g^{-1} = |G|((\mathrm{id}\otimes S)\circ\Delta)(\pi_0)$$

so in the red arrow notation

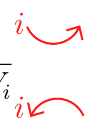
$Z\Bigg($  $\Bigg) = |G|\sum_{i,j}\frac{\dim V_i}{|G|}\frac{\dim V_j}{|G|}$ 

.

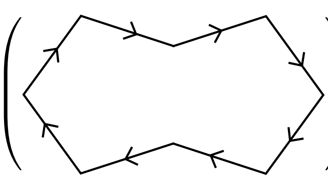
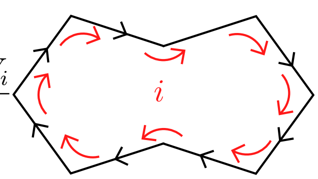
but



is 0 if $i \neq j$ (Schur's lemma) and

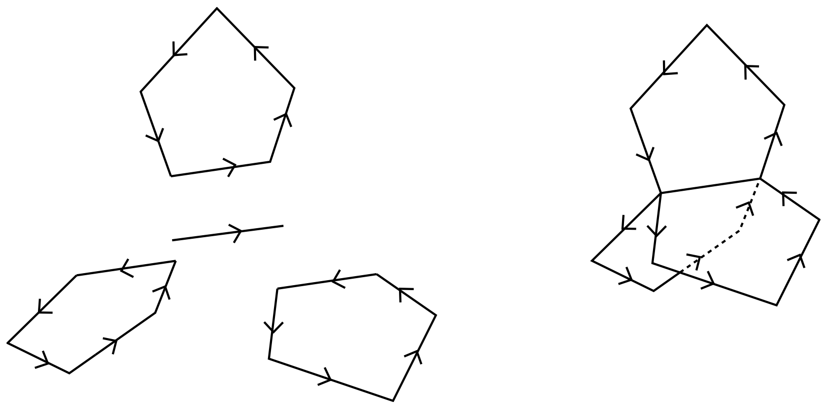
$\frac{1}{\dim V_i}$ 

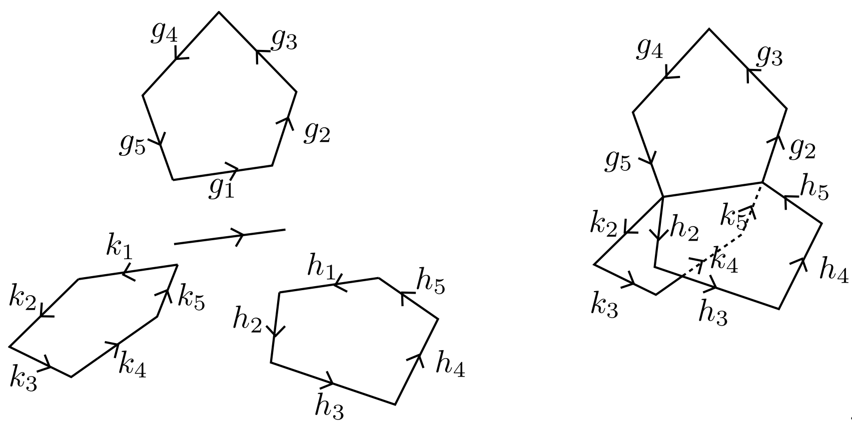
if $i = j$. Therefore

$Z\Bigg($  $\Bigg) = \sum_i \frac{\dim V_i}{|G|}$ 

as it should be.

Similarly, you can count G-labelings on the 2-complex obtained by gluing together three pentagons:

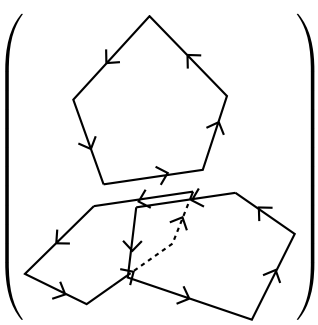




.

Note that the labeling and arrow has been omitted from the “interior” 1-simplex.

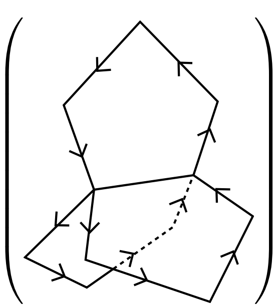
Z



$$= \sum_{\substack{g_1g_2g_3g_4g_5=e \\ h_1h_2h_3h_4h_5=e \\ k_1k_2k_3k_4k_5=e}} (\delta_{g_1} \otimes \cdots \delta_{g_5}) \otimes (\delta_{h_1} \otimes \cdots \otimes \delta_{h_5}) \otimes (\delta_{k_1} \otimes \cdots \otimes \delta_{k_5}).$$

To get the labelings for the three pentagons glued together, you need to sum over the possibilities where $g_1 = h_1^{-1} = k_1^{-1}$. In other words:

Z



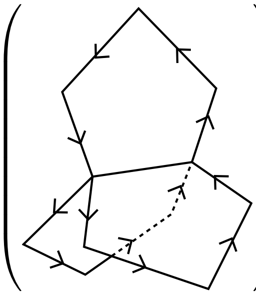
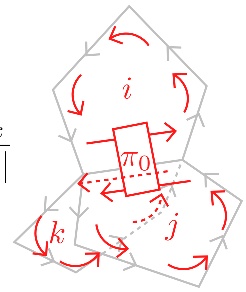
$$= \sum_{\substack{g_1g_2g_3g_4g_5=e \\ h_1h_2h_3h_4h_5=e \\ k_1k_2k_3k_4k_5=e}} \left\langle \sum_g g \otimes g^{-1} \otimes g^{-1}, \delta_{g_1} \otimes \delta_{h_1} \otimes \delta_{k_1} \right\rangle (\delta_{g_2} \otimes \cdots \delta_{g_5})$$

$$\otimes(\delta_{h_2}\otimes\cdots\otimes\delta_{h_5})\otimes(\delta_{k_2}\otimes\cdots\otimes\delta_{k_5}).$$

Since

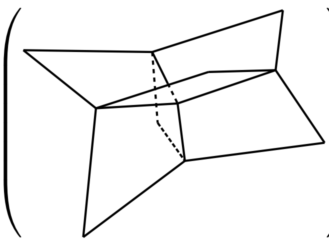
$$\sum_g g\otimes g^{-1}\otimes g^{-1} = |G|((\mathrm{id}\otimes S\otimes S)\circ(\Delta\otimes\mathrm{id})\circ\Delta)(\pi_0)$$

it follows that

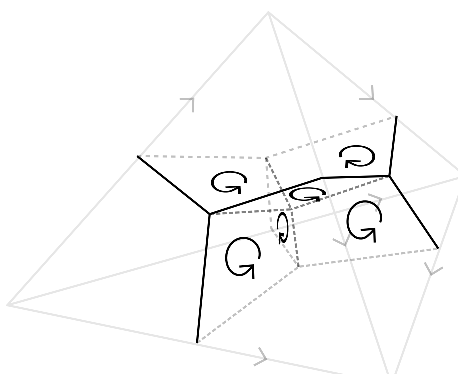
$Z\Bigg($  $\Bigg) = |G|\sum_{i,j,k}\frac{V_i}{|G|}\frac{V_j}{|G|}\frac{V_k}{|G|}$ 

(all the red arrows on a face have the same label i, j or k).

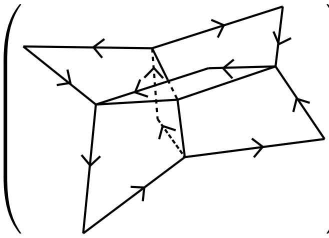
Here is a more complicated complex, the dual triangulation of an ideal 3-simplex:

$Z\Bigg($  $\Bigg)$

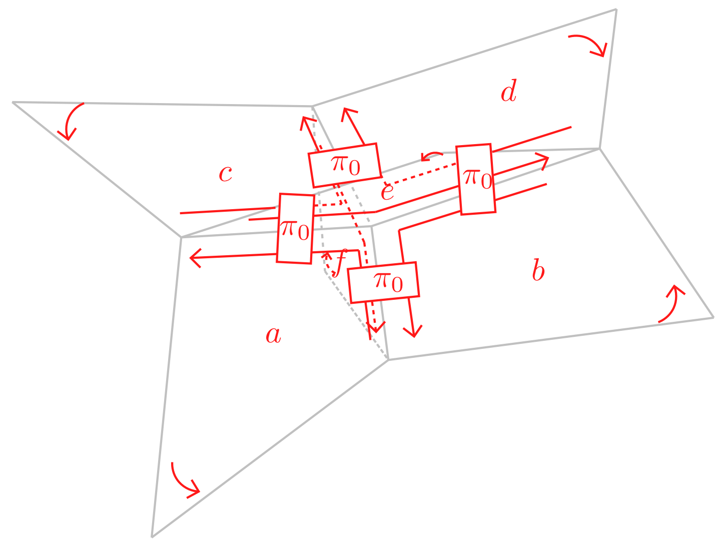
It picks up cyclic orientations on each of its faces from the orienations on edges of the 3-simplex (following a right-hand rule, say):

 .

Orienting the edges of squares following these orientations, then

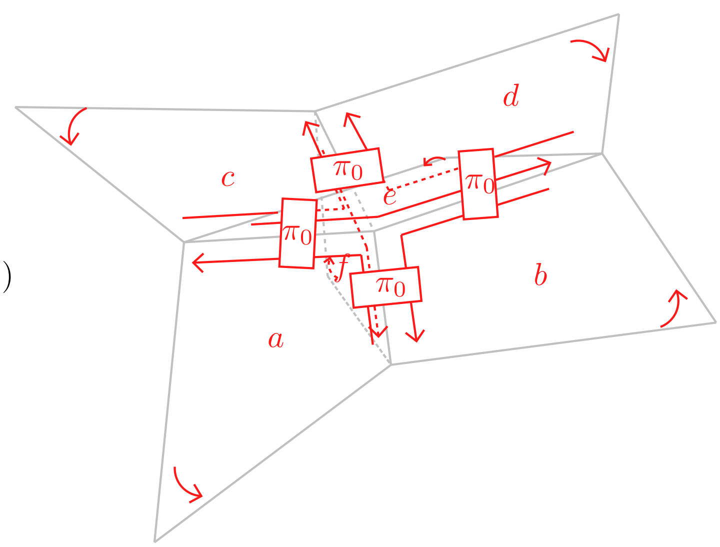
$Z\Bigg($  $\Bigg) =$

$$|G|^4 \frac{1}{|G|} \sum_{a,b,c,d,e,f} \left(\frac{\dim V_a}{|G|} \cdots \frac{\dim V_f}{|G|} \right)$$



(all the arrows on a face labeled by the same index a, b, c, d, e, f). There is one factor of $|G|$ for each edge and one factor of $\frac{1}{|G|}$ for the "interior" vertex. There are six factors of $\frac{1}{|G|}$ inside the sum, each paired with a $\dim V_i$, corresponding to the six faces. One may then write this as

$$= |G|^{-\chi} \sum_{a,b,c,d,e,f} (\dim V_a \cdots \dim V_f)$$



where χ is the Euler characteristic of the complex. The six red arrows around the vertex form what is called a "$6j$ symbol"

Let $\mathcal{I}$ be the set indexing the irreps of G. Let M be a 3-manifold with triangulation τ. Let τ^* denote the dual triangulation. Let $\mathcal{F}$ be the set of functions from faces (i.e., 2-cells) of τ^* into the set $\mathcal{I}$. $\mathcal{F}$ therefore records labels of the faces by elements of $\mathcal{I}$. Given a vertex σ^0 in τ^*, the six labelings incident to that vertex determine a $6j$ symbol as in the previous picture. Call this $6j$ symbol $\langle F, \sigma^0 \rangle$. The interested reader will determine that, upon gluing together all the complexes decorated with red arrows, it follows that

$$\frac{\mathrm{Hom}(\pi_1 M, G)}{|G|} = |G|^{-n} \sum_{F \in \mathcal{F}} \prod_{\text{faces } \sigma^2 \in \tau^*} \dim V_{F(\sigma^2)} \prod_{\text{vertices } \sigma^0 \in \tau^*} \left(\text{contraction of all the } \langle F, \sigma^0 \rangle \right)$$

where n is the number of vertices in τ.