Math 54 Midterm 2

1. $$\frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

2. The dimension of the image is 3, the dimension of the kernel is 0.

3. The ith coordinate of x is $-n+i-1$, so $e_i \cdot x = -n+i-1$.

4. The image of A is the span of the columns, so it's 1-dimensional. The kernel of A is therefore 2-dimensional. The eigenvalues of A are the roots of
$$\det\begin{pmatrix} 2-\lambda & 3 & -1 \\ 4 & 6-\lambda & -2 \\ 0 & 0 & -\lambda \end{pmatrix} = -\lambda^2(\lambda-8)$$
so the eigenvalues are 0 and 8. The 8-eigenspace is the kernel of
$$\begin{pmatrix} 2-8 & 3 & -1 \\ 4 & 6-8 & -2 \\ 0 & 0 & -8 \end{pmatrix}$$
which is the span of $(1,2,0)$. The 0-eigenspace is the kernel of A, a basis of which is $(-3,2,0),\ (0,1,3)$.

5. Since
$$\det(A-\lambda I) = -\lambda(\lambda-1)(\lambda-3)$$
the eigenvalues are $3,1,0$. An eigenbasis consists of a vector in the kernel of
$$\begin{pmatrix} -6 & 0 & 3 \\ -2 & -2 & 1 \\ -6 & 0 & 3 \end{pmatrix}$$
e.g., $(1,0,2)$, a vector in the kernel of
$$\begin{pmatrix} -4 & 0 & 3 \\ -2 & 0 & 1 \\ -6 & 0 & 5 \end{pmatrix}$$
e.g., $(0,1,0)$, and a vector in the kernel of A, e.g., $(1,1,1)$. Therefore if
$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

Then $A = PDP^{-1}$ and

$$A^{100} = PD^{100}P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3^{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3^{100} & 0 & 3^{100} \\ -2 & 1 & 1 \\ -2 \cdot 3^{100} & 0 & 2 \cdot 3^{100} \end{pmatrix}$$

6. The lines are spanned by the vectors $v = (1, 0, 1)$ and $w = (1, -1, 0)$ so you want to know the angle θ between these two vectors. Then

$$\cos(\theta) = \frac{v \cdot w}{\|v\|\|w\|} = \frac{1}{2}$$

so θ is $\pi/3$. If one instead chose v and $-w$, then one would get $\cos(\theta) = -1/2$, so θ would be $2\pi/3$.

7. A basis for this plane is $v_1 = (1, -1, 0)$ and $v_2 = (1, 0, -1)$. Applying Gram-Schmidt to these two vectors works as follows:

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$w_2 = v_2 - (v_2 \cdot u_1)u_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Therefore

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

forms an orthonormal basis. The projection of $(2, 0, 2)$ onto this plane is

$$\left(\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cdot u_1 \right) u_1 + \left(\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cdot u_2 \right) u_2 = \begin{pmatrix} 2/3 \\ -4/3 \\ 2/3 \end{pmatrix}$$

8. U does not have orthonormal columns.