Math 54 Midterm 2 Practice 2 Solutions

1.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

so the inverse is

$$\begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.

$$\det \begin{pmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \end{pmatrix} = -2 \det \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= (-2) \cdot (-1) \cdot \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -2$$

3. $AA^{-1} = 1$ so $1 = \det(AA^{-1}) = \det(A)\det(A^{-1}) = 5\det(A^{-1})$ so $\det(A^{-1}) = 1/5$.

4. $\det(A - \lambda I) = \lambda^4 - 4\lambda^3 + 3\lambda^2 + 4\lambda - 4$. Once you realize that 1 is a root of this polynomial, it can be factored with the help of polynomial long division:

$$\lambda^4 - 4\lambda^3 + 3\lambda^2 + 4\lambda - 4 = (\lambda - 1)(\lambda + 1)(\lambda - 2)^2$$

Therefore the 2-eigenspace is $\ker(A - 2I)$ which has a basis $(1, 0, 0, 1)$, $(0, 0, 1, 0)$. The 1-eigenspace is $\ker(A - I)$ which has a basis $(0, 1, 0, 0)$. The -1-eigenspace is $\ker(A + I)$ which has basis $(1, 1, 1, 0)$. Therefore $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

so that

$$A^{100} = PD^{100}P^{-1} = \begin{pmatrix} (-1)^{100} & 0 & 0 & 2^{100} - (-1)^{100} \\ (-1)^{100} - 1 & 1 & 0 & 1 - (-1)^{100} \\ (-1)^{100} - 2^{100} & 0 & 2^{100} & 2^{100} - (-1)^{100} \\ 0 & 0 & 0 & 2^{100} \end{pmatrix}$$

5. All the side lengths are $\sqrt{2}$, so this is an equilateral triangles (all angles $\pi/3$).

6. The strategy is to find any basis of the plane, then apply Gram-Schmidt to make that basis an orthonormal basis. Row reduction shows that

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

is a basis. Let

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

and let

$$w_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} - \left(\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \cdot u_1 \right) u_1 = \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \\ -1/3 \end{pmatrix}$$

and normalize this to

$$u_2 = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ -2 \\ 3 \\ -1 \end{pmatrix}.$$

(u_1, u_2) forms a basis of the plane. The orthogonal projection of $v = (-1, 1, -1, -1)$ to the plane is $(v \cdot u_1)u_1 + (v \cdot u_2)u_2$, which is $(-4/5, 7/5, -3/5, -4/5)$.

7. If you square this matrix, you get 9 times itself. Therefore its inverse is $1/9$ times itself.

8. Yes, set

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $U\Sigma V^T = U$ is not symmetric.