Math 54 Midterm 2 Practice 1 Solutions

1.
$$\left(\begin{array}{ccc|ccc} 3 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 4 & -3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 0 & -2 & 1 & 0 & 0 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 0 & -2 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 9 & -8 & 4 & 0 & -3 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 3 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -9 & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -6 & -2 \\ 0 & 1 & 0 & 4 & -8 & -3 \\ 0 & 0 & 1 & 4 & -9 & -3 \end{array}\right)$$
so the inverse is
$$\begin{pmatrix} 3 & -6 & -2 \\ 4 & -8 & -3 \\ 4 & -9 & -3 \end{pmatrix}$$

2. For example
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
The only eigenvalues are 0, so the eigenvectors are elements of the kernel. However, the image is clearly 1-dimensional, so the kernel is 2-dimensional. Hence you can't find an eigenbasis.

3. The image is the span of the columns, which is 2-dimensional. Then dimension of the image plus the dimension of the kernel is 3, so the dimension of the kernel is 1.

4. The columns are not linearly independent, so there is an element of the kernel of A. Therefore A is not invertible. Therefore $\det(A) = 0$. The image of this matrix is the line spanned by $(1, 1, 1, 1, 1)$. In particular it sends $(1, 1, 1, 1, 1)$ to $(5, 5, 5, 5, 5)$. Therefore 5 is the only nonzero eigenvalue. It has a nontrivial kernel, so 0 is also an eigenvalue. Since the dimensions of the kernel and image add up to 5, the 5-eigenspace has dimension 1 and the 0-eigenspace has dimension 4.

5.
$$\det(A - \lambda I) = -\lambda(\lambda - 1)^2$$
so the eigenvalues are 0 and 1. Then a basis of
$$\ker(A - I) = \ker \begin{pmatrix} -2 & 2 & 0 \\ -1 & 1 & 0 \\ 3 & -3 & 0 \end{pmatrix}$$

is $(1,1,0)$, $(0,0,1)$. A basis of

$$\ker(A - 0I) = \ker(A)$$

is $(2,1,-3)$. Therefore $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ 3 & -3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Therefore

$$A^{100} = PD^{100}P^{-1} = PDP^{-1} = A$$

6.

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

and the length of this vector is $\sqrt{3^2 + (-1)^2 + 1^2 + 2^2 + 1^2} = 4$.

7. The two vectors coming out of the vertex at $(1,1)$ are

$$v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

and

$$w = \begin{pmatrix} \sqrt{3} \\ -\sqrt{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} - 1 \\ -\sqrt{3} - 1 \end{pmatrix}$$

And $\|v\| = \sqrt{2}$, $\|w\| = \sqrt{8}$, $v \cdot w = 2$. Then

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|} = \frac{1}{2}$$

which implies that the angle at $(1,1)$ is $\pi/3$. The other angles can computed similarly.

This is a 30-60-90 triangle: the side lengths are $\sqrt{2}$, $\sqrt{6}$, $2\sqrt{2}$ with angles $\pi/6$, $\pi/3$, $\pi/2$.

8. The vector $(-2,3,5)$ is orthogonal to all vectors in this plane. Since $\dim(V) + \dim(V^{\perp}) = 3$, then $V^{\perp}$ is the span of $(-2,3,5)$.