Math 54 Midterm 1 Solutions

1.

$$\left(\begin{array}{cccc|c} -2 & 2 & -2 & 1 & 7 \\ - & 1 & 0 & 0 & 1 \\ -4 & 0 & -2 & 1 & 7 \\ -2 & 2 & -1 & 0 & 5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1/2 & -7/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -4 & 2 & -1 & -7 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1/2 & -5/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1/2 & -5/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & 0 & -1/2 & -1/2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

so the solution is $(-1, 1, -1, 1)$.

2. (a) plane
   (b) plane
   (c) line
   (d) point

3.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

4. (a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (b)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5. For example

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. If it were, then there would be a solution to

$$a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

One can row reduce to solve this equation:

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right)$$

which doesn't have a solution.

6. (a) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two vectors. Then $5\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, so the vectors are linearly dependent.

   (b) If these vectors are linearly dependent, then there should be a solution (a_1, a_2, a_3) to

$$a_1 \begin{pmatrix} -2 \\ 2 \\ 1 \\ -4 \end{pmatrix} + a_2 \begin{pmatrix} 4 \\ 5 \\ 1 \\ 2 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   To solve this, row reduce

$$\left(\begin{array}{ccc|c} -2 & 4 & 0 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ -4 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   to get

$$a_1 + \frac{2}{3}a_3 = 0$$

$$a_2 + \frac{1}{3}a_3 = 0$$

   Therefore there are many solutions. For example $a_1 = -2$, $a_2 = -1$, $a_3 = 3$ is a solution. So

$$-2 \begin{pmatrix} -2 \\ 2 \\ 1 \\ -4 \end{pmatrix} + -1 \begin{pmatrix} 4 \\ 5 \\ 1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   so the three vectors are linearly dependent.

7. Neither of the planes is a subspace of $\mathbb{R}^3$ since neither contains 0:

$$\begin{pmatrix} t+s \\ s+1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has no solution (t, s) and

$$\begin{pmatrix} 2s \\ s-t-1 \\ t+s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has no solution (t, s).

The two planes do no intersect. The first plane is the plane $x - y - z = -1$ and the second is the plane $x - y - z = 1$. To see this, eliminate t and s from

$$x = t + s$$
$$y = s + 1$$
$$z = t$$

to get a relation between x, y, and z for the first plane, and also eliminate t and s from

$$x = 2s$$
$$y = s - t - 1$$
$$z = t + s$$

to get a relation between x, y, and z for the second plane. It is easy to see that

$$\begin{cases} x - y - z = -1 \\ x - y - z = 1 \end{cases}$$

does not have any solutions, so the planes do not intersect.

8. (a) f is not surjective, since the image only contains points whose x and z coordinates are the same. f is injective, however. If (x, y) is in the kernel of f, then

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which implies that $x = 0$ and $y = 0$. Therefore the kernel of f is just $\{(0, 0)\}$, so f is injective.

(b) f is both surjective and injective. It can be represented by a matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the columns of this matrix span the image of f. The two columns are two noncolinear vectors in $\mathbb{R}^2$ and thus form a basis of $\mathbb{R}^2$. In particular, their span is all of $\mathbb{R}^2$. Hence the image of f is all of $\mathbb{R}^2$. To see that f is injective, compute the kernel of f. This is the set of pairs (x_1, x_2) such that $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$. The only such pair is $(0, 0)$. Hence the kernel of f is just $\{(0, 0)\}$. Hence f is injective.