Math 54 Midterm 1 Practice 2 Solutions

1.

$$\left(\begin{array}{cccc|c} 2 & -2 & -1 & -1 & -5 \\ 1 & 0 & 0 & -1 & 0 \\ -2 & 0 & 1 & 2 & 0 \\ 4 & -2 & -2 & -4 & -6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 2 & -2 & -1 & -1 & -5 \\ -2 & 0 & 1 & 2 & 0 \\ 2 & -1 & -1 & -2 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & -2 & -1 & 1 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & -5 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

so the system of equations is equivalent to the system

$$\begin{cases} x_1 = 1 \\ x_2 = 3 \\ x_3 = 0 \\ x_4 = 1 \end{cases}$$

therefore the solution is $(1, 3, 0, 1)$.

2. (a) line (corrected)
   (b) point or a line, depending on if the equations are viewed with 2 or 3 variables (corrected)
   (c) point
   (d) no solution

3. (a) Want to solve

$$a_1 \begin{pmatrix} 1 \\ -5 \\ -4 \\ 8 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 9 \\ 8 \\ -16 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for a_1, a_2, and a_3. This amounts to row reducing

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -5 & 9 & -1 & 0 \\ -4 & 8 & -1 & 0 \\ 8 & -16 & 2 & 0 \end{array}\right)$$

which (after some steps) row reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so the solutions satisfy

$$\begin{cases} a_1 = \frac{1}{4}a_3 \\ a_2 = \frac{1}{4}a_3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

for example $(1, 1, 4)$ is a solution. Then

$$\begin{pmatrix} 1 \\ -5 \\ -4 \\ 8 \end{pmatrix} + \begin{pmatrix} -1 \\ 9 \\ 8 \\ -16 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so these vectors are not linearly independent.

(b) Want to solve

$$a_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -4 \\ 0 \\ 2 \\ 4 \end{pmatrix} + a_3 \begin{pmatrix} -5 \\ -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for a_1, a_2, and a_3. This amounts to row reducing

$$\left(\begin{array}{ccc|c} 0 & -4 & -5 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 4 & 1 & 0 \end{array}\right)$$

which (after some steps) row reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so the solutions satisfy

$$\begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ 0 = 0 \end{cases}$$

therefore the only solution is $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent.

(c) Similar to the previous two, the row reduction

$$\left(\begin{array}{ccc|c} 1 & 1 & -9 & 0 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

implies that the only solutions to

$$a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} -9 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

are $a_1 = a_2 = a_3 = 0$. Therefore the vectors are linearly dependent.

4. Let's row reduce. The columns corresponding to the pivot variables will form a basis of the image.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so all the variables are pivot variables. Therefore a basis for the image is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Since there are four of these vectors they form a basis for $\mathbb{R}^4$, not just for some smaller subspace of $\mathbb{R}^4$. Therefore the image is all of $\mathbb{R}^4$.

Determining injectivity involves computing the kernel. If the kernel is $\{0\}$ then the map is injective. Otherwise it is not injective. Computing the kernel involves solving

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for (x_1, x_2, x_3, x_4). Solving this in turn involves row reducing

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

This is the same row reduction as before (only with the matrix augmented by a column of zeros), so this row reduces to

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

implying that the only element in the kernel is $(0, 0, 0, 0)$. Therefore the map is injective.

5. A basis for the image is given the columns corresponding to pivot variables:

$$\begin{pmatrix} 2 & -4 & 6 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first two variables are the pivot variables, so a basis for the image consists of

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

The kernel is the set of solutions to the equation

$$\begin{pmatrix} 2 & -4 & 6 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which in turn involves the same row reduction, only augmented by a column of zeros:

$$\left(\begin{array}{cccc|c} 2 & -4 & 6 & 4 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since there are two free variables (x_3 and x_4), the solution to this system is a plane. You can write x_1 and x_2 in terms of x_3 and x_4:

$$x_1 = -x_3$$

$$x_2 = x_3 + x_4$$

so the elements in the kernel are those vectors of the form

$$\begin{pmatrix} -x_3 \\ x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

i.e., those of the form

$$x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the vectors

$$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

span the kernel. Since they are linearly independent (there are two of them and they're not multiples of one another) they form a basis of the kernel.

6. A vector in the kernel is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(there are many other vectors you could pick, for example $(1, -1, 0, 0, 0)$). Determining the kernel of this map involves row reducing

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

which becomes

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

There are four free variables so the kernel is a 4-dimensional linear subspace of $\mathbb{R}^5$.

7.
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & \frac{5}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 \end{array}\right)$$

so that the solution set is equivalent to the equations

$$\begin{cases} x_1 + \frac{5}{3}x_3 - \frac{2}{3}x_4 = 0 \\ x_2 + \frac{2}{3}x_3 + \frac{1}{3}x_4 = 0 \end{cases}$$

express the pivot variables in terms of the free variables:

$$x_1 = -\frac{5}{3}x_3 + \frac{2}{3}x_4$$

$$x_2 = -\frac{2}{3}x_3 - \frac{1}{3}x_4$$

so that the points in the solution plane are of the form

$$\begin{pmatrix} -\frac{5}{3}x_3 + \frac{2}{3}x_4 \\ -\frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

for some values of x_3 and x_4. These points, equivalently, are

$$x_3 \begin{pmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$$

so the two vectors

$$\begin{pmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$$

span the plane. Since they are linearly independent (they are two vectors and one is not a multiple of the other) they form a basis for the plane.

8. You have to check where the eight vertices of the cube C are mapped. The vertices are the eight vectors whose entries are either 0 or 1. These are mapped to

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Therefore the vertices of C are mapped to the vertices of a hexagon (call it H) with corners at the six outermost of these vectors:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

as well as a point at the center of H. Since linear maps map line segments to line segments, the edges of the cube are mapped to line segments between the vertices of H and the central point. When you draw these line segments in H, you can see that this linear map squashes the cube C down to the hexagon H.