Math 54 Midterm 1 Practice 1 Solutions

1.
$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 1 & 4 & -1 & 2 & 3 \\ -2 & -1 & 0 & -2 & -4 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 3 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right)$$
$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 3 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Therefore this system is equivalent to the system

$$\begin{cases} x_1 = 2 \\ x_2 = 0 \\ x_3 = -1 \\ x_4 = 0 \end{cases}$$

so the solution set is the point $(2, 0, -1, 0)$.

2. (a) line
   (b) line (corrected)
   (c) point
   (d) line (corrected)

3. (a) From left to right, call the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ so they form a linearly dependent set.
   (b) From left to right, call the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, so they form a linearly dependent set.
   (c) These are three vectors that lie in a plane so there should be some linear relation between them. In fact, calling them $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then
$$\frac{7}{2}\mathbf{v}_1 + \frac{1}{7}\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_3 = \mathbf{0}$$
   (d) These are linearly independent. They are two vectors and they are not multiples of one another.

4. An easy way is to add constants to the equations defining the planes. For example, you could take the plane given by

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

and the plane given by

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases}$$

These planes have no intersection since points in the first must have x_1-coordinate equal to 1 and points in the second must have x_1-coordinate equal to 0.

5. Suppose they intersect at the point parametrized by t in the first line and the point parametrized by s in the second line. Then

$$\begin{pmatrix} 1 \\ 2t+1 \\ t+1 \end{pmatrix} = \begin{pmatrix} 2s+7 \\ s-6 \\ 2s+2 \end{pmatrix}$$

the solution of which is the following system of three equations in two variables

$$\begin{cases} -2s = 7 \\ 2t - s = -7 \\ t - 2s = 1 \end{cases}$$

One can solve this to find a single solution at $t = -5$, $s = -3$, so the two lines do intersect (at $(1, -9, -4)$).

6. Note that x can be expressed in terms of y and z: $x = -2y - 3z$. Therefore the set of vectors in the plane in the set of vectors of the form $(-2y - 3z, y, z)$ (as y and z range over all real numbers). Therefore any vector in the plane can be written as

$$y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

for some y and z. Therefore the two vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

span the plane. These two vectors are linearly independent (if they weren't, they would be multiples of one another), so they form a basis for the plane $x + 2y + 3z = 0$.

7. f is not surjective, since the image only consists of vectors with second coordinate equal to 0. f is not injective since $(-1, 1, 0)$ and $(0, 0, 0)$ both are mapped to $(0, 0)$.

8. The image is the span of the columns. In this case the columns are all multiples of each other, so their span is the line spanned by $(1, 3)$. Equivalently, this is the line $3x - y = 0$ in $\mathbb{R}^2$.