Math 54
Terminology from Set Theory

A set is a collection of things, usually mathematical things. For example the real numbers $\mathbb{R}$ form a set. Curly brackets $\{\}$ mean "the set containing", for example

$$\{0, 1, 2\}$$

is the set containing 0, 1, and 2. Other examples of sets are $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, etc.

If X is a set the sentence "x is an element of X" or, equivalently, "x is in X" is denoted by $x \in X$.

Recall from high school that a function is something that takes as input a number and outputs another number. For example $f(x) = x^2$ defines a function whose output is the square of its input. $f(3) = 9$ means that 3 is input into the function, which then outputs $3^2 = 9$.

Let X and Y be sets. A map of sets f from X to Y, denoted $f : X \to Y$, is something that takes as input elements of X and outputs elements of Y. For example

$$f : \{0, 1, 2\} \to \{0, 1\}$$

$$f(0) = 0,\ f(1) = 1,\ f(2) = 1$$

is a map of sets from $\{0, 1, 2\}$ to $\{0, 1\}$. It takes 0 to 0, 1 to 1, and 2 to 1.

The functions you know from high school are maps of sets. For example, $\log : \mathbb{R}_{>0} \to \mathbb{R}$.

If $f : X \to Y$ is a map of sets, then we call X the "domain" of f and Y the "codomain" of f. The set of outputs of f is called the "image" of f. Note that the codomain and the image are not necessarily the same thing. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, then the codomain is $\mathbb{R}$ whilst the image is $\mathbb{R}_{\geq 0}$.

We say f is "surjective" if every element in the codomain is an output. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, then f is not surjective, since the negative numbers in the codomain are not outputs of f. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then it is surjective. Some people also say "onto" as a synonym for "surjective".

We say f "injective" if every output corresponds to precisely one input. If $f(x) = x^2$ then f is not injective since the output 1 corresponds to the inputs -1 and 1. If instead $f(x) = x^3$ then each output has a single input. Some people also say "one-to-one" as a synonym for "injective".

A map $f : X \to Y$ is "bijective" if it is both injective and surjective. This means that every element in Y is the output of precisely one input in X. Said another way, for each $y \in Y$, there exists one element $x \in X$ such that $f(x) = y$. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then, as noted above, f is both injective and surjective and therefore bijective. Given $y \in \mathbb{R}$, there is precisely one element $x \in \mathbb{R}$ such that $f(x) = y$, namely $\sqrt[3]{y}$.

If $f : X \to Y$ is bijective, because there is precisely one input corresponding to each element in the codomain, one can define an inverse function $f^{-1} : Y \to$

X such that $f^{-1}(f(x)) = x$. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$. For this reason, "invertible" is a synonym for "bijective".

This course concerns itself with certain maps $f : \mathbb{R}^n \to \mathbb{R}^m$. These maps are required to satisfy two properties:

$$f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}),\ \ \vec{v}, \vec{w} \in \mathbb{R}^n$$

$$f(c\vec{v}) = cf(\vec{v}),\ \ \vec{v} \in \mathbb{R}^n,\ \ c \in \mathbb{R}$$

Maps that satisfy these two properties are called "linear". Note that these two properties correspond to the two important operations on $\mathbb{R}^n$: addition of vectors, and scalar multiplication. The two properties can be summarized by saying that "adding and then applying f is the same as applying f and then adding" and "scaling and then applying f is the same as applying f and then scaling".

Here is an example of linear map $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 5x + 2y.$$

Here is an example of something that is not a linear map

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 5x + 2y + 1$$

For example the latter map does not satisfy the second property of a linear map when $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c = 2$.

Linear maps are often called linear transformations. Most of the time, and throughout this course, the words "function", "map" and "transformation" mean the same thing.